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# Sturm-Liouville Series of Normal Functions in the Theory of Integral Equations

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# IV. *Sturm-Liouville Series of Normal Functions in the Theory of Integral Equations.*

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## *Introduction.*

One of the most important branches of the theory of integral equations is connected with the problem of representing a function as a series of normal functions, Hilbert* and Schmidt,† who made the earliest contributions, have been able to obtain sufficient conditions under which an assigned function may be expanded in terms of a system of normal functions belonging to a symmetric characteristic function (*kern*). These conditions are narrow in respect to the nature of the function which may be expanded, but they have the advantage of applying to very general systems of normal functions. They apply, in particular, to the expansion of a function in both the sine and cosine series of Fourier. It is in the light of our knowledge of the properties of the latter series that the narrowness above referred to becomes evident. In point of fact, the Hilbert-Schmidt theory is only applicable to Fourier's series corresponding to a function which has a continuous second differential coefficient in $(0, \pi)$, and which furthermore satisfies certain boundary conditions at the end points of the interval. For example, in the case of the sine series, the function must vanish at both end points. It would appear, therefore, that the wide generality as to the system of normal functions is obtained at the cost of the generality of the function which it is desired to represent.

Later memoirs by Kneser and Hobson‡ have made it abundantly clear that, by restricting the nature of the system of normal functions, results may be obtained in regard to the representation of very much wider classes of functions than were

* 'Gött. Nachr.' (1904), pp. 71–78.

† 'Math. Ann.,' vol. 63, pp. 451–453.

‡ See also A. C. Dixon, 'Proc. Lond. Math. Soc.,' series 2, vol. 3, pp. 83–103.

contemplated by HILBERT and SCHMIDT. KNESER'S paper* is of importance as marking the first step in this direction, but his results are far less general than those obtained by HOBSON and published last year in the 'Proceedings of the London Mathematical Society.'† As one of many interesting applications of a general convergence theorem, the latter‡ has been able to show that any Sturm-Liouville series corresponding to an assigned function converges at a point, provided that the function has a Lebesgue integral in the interval of representation, and is of limited total fluctuation in an arbitrarily small neighbourhood of the point in question. Taken in conjunction with other results of a similar kind, this cannot fail to suggest the possibility of extending most of the well-known theorems on FOURIER'S series to the whole class of Sturm-Liouville expansions. It is the purpose of this memoir to show that all the more important theorems are capable of this extension.

It will not be necessary to give here a detailed account of the results obtained, seeing that those of importance have from time to time been summarised as formal theorems printed in italics. It may, however, be useful to say a few words as to the plan on which the memoir has been written. The first section is devoted to the proof of two theorems of convergence which find repeated applications in the sequel. In § 4 of the second, a theorem relative to the expansion of a function as a series of normal functions is established. The theorem has reference to a very wide class of expansions. The only obstacle which can hinder its application in any given case is the difficulty of determining an asymptotic formula for the solving function $K_\lambda(s, t)$, when $\lambda$ is negative and numerically large. At the commencement of the third section a formula of this kind is obtained which makes it possible to apply the theorem to what I have called canonical Sturm-Liouville series (III., § 20). The latter portion of the section is devoted to this application. The results obtained are extended so as to apply to the most general form of Sturm-Liouville series. The fourth, and remaining, section is given up to a discussion of questions of convergence. It is here that the properties of orthogonal functions, proved in the latter half of the second section, find their application.

In conclusion, I may say that in later memoirs I hope to further develop the ideas which have here been made use of. With such modifications as are necessary from the fact that the characteristic function is no longer limited, I hope especially to apply them to expansions in LEGENDRE'S and in BESSEL'S functions.

## I.—THE THEOREMS OF CONVERGENCE.

§ 1. In the following pages we shall find it necessary to make frequent use of two theorems of convergence. These properly belong to the general theory of series (or

* 'Math. Ann.,' vol. 63, pp. 477–524.

† Series 2, vol. 6, pp. 349–395.

‡ Pp. 386–387.



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sequences), and, apart from their applications, have no connection with the theory developed below. It will, therefore, be convenient to enunciate and prove them at the outset.

The first theorem is :—

*Let $g(s, n)$ be a function defined for all values of s in the interval $(a, b)$, and for a set of values of n which have $+\infty$ for an improper limiting point. Let this function be such that* (1) *the upper limit of* $|g(s, n)|$ *is a finite number* $\bar{g}$, *and* (2) $\int_a^b g(s, n)\, ds$ *exists as a Lebesgue integral for each value of n. Let $g(s, n)$ be related to a limited function, $g(s)$, defined in $(a, b)$ in such a way that*

$$\lim_{n \to \infty} g(s, n) = g(s)$$

*either at each point of $(a, b)$ or at those points of $(a, b)$ which do not belong to a certain set of zero measure. Then, if $f(s)$ is any function which possesses a Lebesgue integral in $(a, b)$, we have*

$$\lim_{n \to \infty} \int_a^b g(s, n) f(s)\, ds = \int_a^b g(s) f(s)\, ds.$$

To prove this, let us take any positive number $\eta$. Let us denote by $j_n$ the set of points of $(a, b)$ at which

$$|g(s, n) - g(s)| > \eta,*$$

and by $J_n$ the set complementary to $j_n$. Then we have

$$\int_a^b g(s, n) f(s)\, ds - \int_a^b g(s) f(s)\, ds = \int_{j_n} + \int_{J_n} [g(s, n) - g(s)] f(s)\, ds.$$

The numerical value of the first integral is evidently not greater than

$$(\bar{g} + g') \int_{j_n} |f(s)|\, ds,$$

where $g'$ is the upper limit of $|g(s)|$ in $(a, b)$, and the numerical value of the second is not greater than

$$\eta \int_{J_n} |f(s)|\, ds,$$

which is not greater than

$$\eta \int_a^b |f(s)|\, ds.$$

We thus have the inequality

$$\left| \int_a^b g(s, n) f(s)\, ds - \int_a^b g(s) f(s)\, ds \right| \leq (\bar{g} + g') \int_{j_n} |f(s)|\, ds + \eta \int_a^b |f(s)|\, ds.$$

* It should be observed that, in virtue of its relation to the functions $g(s, n)$, $g(s)$ is summable, and that $j_n$ is therefore measurable.

Now, if $\epsilon$ is an arbitrarily assigned positive number, we can choose $\eta$ so small that

$$\eta \int_a^b |f(s)|\, ds < \tfrac{1}{2}\epsilon.$$

Further, the measure of $j_n$ tends to zero when $n$ increases indefinitely, in virtue of our hypothesis as to the relation between $g(s, n)$ and $g(s)$; hence

$$\lim_{n \to \infty} \int_{j_n} |f(s)|\, ds = 0.$$

We can, therefore, find a number N such that

$$(\bar{g}+g') \int_{j_n} |f(s)|\, ds < \tfrac{1}{2}\epsilon,$$

for all values of $n \geqq \mathrm{N}$. It follows that, for $n \geqq \mathrm{N}$,

$$\left| \int_a^b g(s, n) f(s)\, ds - \int_a^b g(s) f(s)\, ds \right| < \epsilon,$$

which establishes the theorem.

§ 2. A theorem corresponding to that of the preceding paragraph holds for functions of two or more independent variables. The proof in each case follows the same lines as that which has just been given. It will, therefore, be sufficient to state the theorem for two independent variables :—

*Let $g(s, t, n)$ be a function defined at all points of the square, Q, which consists of the points for which $a \leqq s \leqq b$, $a \leqq t \leqq b$, and for a set of values of $n$ which have $+\infty$ for an improper limiting point. Let the function be such that* (1) *the upper limit of $|g(s, t, n)|$ is a finite number $\bar{g}$, and* (2) $\int_{(Q)} g(s, t, n)\,(ds\, dt)$ *exists as a Lebesgue integral for each value of $n$. Let $g(s, t, n)$ be related to a limited function $g(s, t)$ defined in Q in such a way that*

$$\lim_{n \to \infty} g(s, t, n) = g(s, t)$$

*either at each point of Q, or at those points of it which do not belong to a certain set of zero measure. Then, if $f(s, t)$ is any function which possesses a Lebesgue integral in Q, we have*

$$\lim_{n \to \infty} \int_{(Q)} g(s, t, n) f(s, t)\,(ds\, dt) = \int_{(Q)} g(s, t) f(s, t)\,(ds\, dt).$$

If $f(s)$ possesses a Lebesgue integral in $(a, b)$, $f(s) f(t)$ possesses a Lebesgue integral in Q. It follows that, with the hypothesis of the theorem just enunciated,

$$\lim_{n \to \infty} \int_{(Q)} g(s, t, n) f(s) f(t)\,(ds\, dt) = \int_{(Q)} g(s, t) f(s) f(t)\,(ds\, dt).$$

§ 3. The second theorem, to which reference was made above, is :—

*Let $g(s, t, n)$ be a function defined at all points of the rectangle, R, which consists of the points for which $a_1 \leq s \leq b_1$, $a \leq t \leq b$, and for a set of values of $n$ which have $+\infty$ for an improper limiting point. Let this function be such that* (1) *the upper limit of* $|g(s, t, n)|$ *is a finite number* $\bar{g}$, *and* (2) $\int_a^b g(s, t, n)\, dt$ *exists as a Lebesgue integral for all values of $s$ and $n$. Let $g(s, t, n)$ be related to a limited function $g(s, t)$ defined in R in such a way that, as $n$ tends to $\infty$, $g(s, t, n)$ converges uniformly to $g(s, t)$ either in the whole of R, or, for each positive number $\eta$, in those parts of it which correspond to*

$$|t - \alpha_m s - \beta_m| \geq \eta, \qquad (m = 1, 2, \ldots, \mathrm{M}),$$

*where the numbers $\alpha_1, \alpha_2, \ldots, \alpha_{\mathrm{M}}$ are all finite. Then, if $f(t)$ is any function which has a Lebesgue integral in $(a, b)$,*

$$\int_a^c g(s, t, n) f(t)\, dt$$

*converges uniformly to*

$$\int_a^c g(s, t) f(t)\, dt,$$

*for $a \leq c \leq b$, $a_1 \leq s \leq b_1$, as $n$ tends to $\infty$.*

In the first place, let us assume that $g(s, t, n)$ converges uniformly to $g(s, t)$ in the whole of R. When any positive number $\epsilon$ is assigned, we can then choose N great enough to ensure that at each point of R

$$|g(s, t, n) - g(s, t)| < \epsilon \Big/ \int_a^b |f(t)|\, dt$$

for all values of $n \geq \mathrm{N}$. From this we see at once that

$$\left| \int_a^c g(s, t, n) f(t)\, dt - \int_a^c g(s, t) f(t)\, dt \right| < \epsilon \int_a^c |f(t)|\, dt \Big/ \int_a^b |f(t)|\, dt,$$

that is to say,

$$< \epsilon,$$

for $a \leq c \leq b$, $a_1 \leq s \leq b_1$, and $n \geq \mathrm{N}$. The theorem is thus proved for this case.

More generally, let us suppose that, for each positive number $\eta$, $g(s, t, n)$ converges uniformly to $g(s, t)$ in those parts of R which correspond to

$$|t - \alpha_m s - \beta_m| \geq \eta, \qquad (m = 1, 2, \ldots, \mathrm{M}); \quad \ldots \ldots \quad (1)$$

then, selecting any value of $\eta$, for each pair of values of $c$ and $s$ the points of $a \leq t \leq c$ which do not satisfy *all* the inequalities just written lie in a set $(j_{c,s})$

of intervals* whose number does not exceed M. Let $I_{c,s}$ be the greatest of the values of $\int |f(t)| dt$ in the various intervals of $j_{c,s}$. Clearly we have

$$\left| \int_{j_{c,s}} [g(s, t, n) - g(s, t)] f(t) \, dt \right| \leq (\bar{g} + g') \mathrm{M} \mathrm{I}_{c,s},$$

where $g'$ is the upper limit of $|g(s, t)|$ in the rectangle R. Hence, if $J_{c,s}$ is portion of the interval $a \leq t \leq c$ which is not covered by one or other of the intervals $j_{c,s}$.

$$\left| \int_a^c g(s, t, n) f(t) \, dt - \int_a^c g(s, t) f(t) \, dt \right| \leq \left| \int_{J_{c,s}} [g(s, t, n) - g(s, t)] f(t) \, dt \right| + (\bar{g} + g') \mathrm{M} \mathrm{I}_{c,s}. \quad (2)$$

Now a Lebesgue integral is a continuous function of its upper limit; hence, if $\epsilon$ is an arbitrarily assigned positive number, we can choose $\eta$ so small that the Lebesgue integral of $|f(t)|$ in any interval of length $2\eta$, which lies in $(a, b)$, is less than

$$\frac{\epsilon}{2(\bar{g} + g')\mathrm{M}}.$$

With this choice of $\eta$ we have

$$(\bar{g} + g') \mathrm{M} \mathrm{I}_{c,s} < \tfrac{1}{2}\epsilon$$

for all values of $c$ in $(a, b)$ and of $s$ in $(a_1, b_1)$, since $I_{c,s}$ is the value of

$$\int |f(t)| \, dt$$

in an interval of length not greater than $2\eta$.

Again, in virtue of our hypothesis as to the uniform convergence of $g(s, t, n)$, we can find a number N great enough to ensure that

$$|g(s, t, n) - g(s, t)| < \epsilon \Big/ 2 \int_a^b |f(t)| \, dt$$

for all values of $s$ and $t$ satisfying (1), and for all values of $n \geq \mathrm{N}$. Since the points of $J_{c,s}$ satisfy (1) for each pair of values of $s$ and $c$, we have

$$\left| \int_{J_{c,s}} [g(s, t, n) - g(s, t)] f(t) \, dt \right| < \epsilon \int_{J_{c,s}} |f(t)| \, dt \Big/ 2 \int_a^b |f(t)| \, dt,$$

which is

$$< \tfrac{1}{2}\epsilon$$

for all values of $c$ in $(a, b)$, of $s$ in $(a_1, b_1)$, and of $n \geq \mathrm{N}$. It follows from (2) that, for these values of $c$, $s$, and $n$,

$$\left| \int_a^c g(s, t, n) f(t) \, dt - \int_a^c g(s, t) f(t) \, dt \right| < \epsilon,$$

which establishes the general theorem.

* These intervals may, of course, overlap.

As a corollary to this theorem it should be noticed that, with the same hypotheses,

$$\int_a^b g(s, t, n) f(t)\, dt$$

converges uniformly to

$$\int_a^b g(s, t) f(t)\, dt,$$

for values of $s$ in $(a_1, b_1)$, as $n$ tends to $\infty$.

§ 4. The theorem we have just proved, like that of § 1, admits of generalisation. On account of its importance in what follows, we state the theorem :—

*Let* $g(s, t, u, n)$ *be a function defined at all points of the rectangular parallelopiped,* P, *which consists of the points* $a_2 \leq s \leq b_2$, $a_1 \leq t \leq b_1$, $a \leq u \leq b$, *and for a set of values of* $n$ *which have* $+\infty$ *for an improper limiting point. Let this function be such that* (1) *the upper limit of* $|g(s, t, u, n)|$ *is finite, and* (2) $\int_a^b g(s, t, u, n)\, du$ *exists as a Lebesgue integral for all values of* $s$, $t$, *and* $n$. *Let* $g(s, t, u, n)$ *be related to a limited function* $g(s, t, u)$ *defined in the whole of* P *in such a way that, as* $n$ *tends to* $\infty$, $g(s, t, u, n)$ *converges uniformly to* $g(s, t, u)$, *either in the whole of* P *or, for each positive number* $\eta$, *in those parts of it which correspond to*

$$|u - \alpha_m s - \beta_m t - \gamma_m| \geq \eta, \qquad (m = 1, 2, \ldots, \mathrm{M}),$$

*where the numbers* $\alpha_m$, $\beta_m$ *are all finite. Then, if* $f(u)$ *is any function which has a Lebesgue integral in* $(a, b)$,

$$\int_a^c g(s, t, u, n) f(u)\, du$$

*converges uniformly to*

$$\int_a^c g(s, t, u) f(u)\, du$$

*for* $a \leq c \leq b$, $a_1 \leq t \leq b_1$, $a_2 \leq s \leq b_2$, *as* $n$ *tends to* $\infty$.

In particular, we see that

$$\int_a^b g(s, t, u, n) f(u)\, du$$

converges uniformly to

$$\int_a^b g(s, t, u) f(u)\, du$$

in the rectangle $a_1 \leq t \leq b_1$, $a_2 \leq s \leq b_2$.

Again, let us suppose that $a = a_2$, $b = b_2$; then, writing $c = s$, we see that, as $n$ tends to $\infty$,

$$\int_a^s g(s, t, u, n) f(u)\, du$$

converges uniformly to

$$\int_a^s g(s, t, u) f(u)\, du$$

for $a \leq s \leq b$, $a_1 \leq t \leq b_1$. Also it will be seen, without difficulty, that

$$\int_s^b g(s, t, u, n) f(u)\, du$$

converges uniformly to

$$\int_s^b g(s, t, u) f(u)\, du$$

for these values of $s$ and $t$.

Retaining the hypothesis that $a = a_2$, $b = b_2$, it is evident that

$$G(s, t, v, n) = \int_v^s g(s, t, u, n) f(u)\, du,$$

being equal to

$$\int_a^s g(s, t, u, n) f(u)\, du - \int_a^v g(s, t, u, n) f(u)\, du,$$

converges uniformly to

$$G(s, t, v) = \int_v^s g(s, t, u) f(u)\, du$$

for $a \leq s \leq b$, $a_1 \leq t \leq b_1$, $a \leq v \leq b$. It follows at once that, as $n$ tends to $\infty$,

$$\int_a^b dv \int_v^s g(s, t, u, n) f(u)\, du$$

converges uniformly to

$$\int_a^b dv \int_v^s g(s, t, u) f(u)\, du$$

for $a \leq s \leq b$, $a_1 \leq t \leq b_1$.

## II.—General Theorems Relative to the Expansion of an Arbitrary Function as a Series of Normal Functions. Fundamental Properties of Systems of Normal Functions.

§ 1. Let $\kappa(s, t)$ be a function of positive type defined in the square $a \leq s \leq b$, $a \leq t \leq b$. Let $\psi_1(s)$, $\psi_2(s)$, ..., $\psi_n(s)$, ..., be a complete system of normal functions of $\kappa(s, t)$, corresponding to singular values $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$. It has been shown* that the series

$$\frac{\psi_1(s)\, \psi_1(t)}{\lambda_1} + \frac{\psi_2(s)\, \psi_2(t)}{\lambda_2} + \ldots + \frac{\psi_n(s)\, \psi_n(t)}{\lambda_n} + \ldots$$

is absolutely and uniformly convergent, and that its sum function is $\kappa(s, t)$. More generally, the series

$$\frac{\psi_1(s)\, \psi_1(t)}{\lambda_1 - \lambda} + \frac{\psi_2(s)\, \psi_2(t)}{\lambda_2 - \lambda} + \ldots + \frac{\psi_n(s)\, \psi_n(t)}{\lambda_n - \lambda} + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* "Functions of Positive and Negative Type and their Connection with the Theory of Integral Equations," 'Phil. Trans. Roy. Soc.,' A, vol. 209, pp. 439–446.

converges absolutely and uniformly, and has for its sum function $K_\lambda(s, t)$, the solving function of $\kappa(s, t)$.

Let $f(s)$ be a function which has a Lebesgue integral in $(a, b)$. Since (1) is uniformly convergent in the square $a \leq s \leq b$, $a \leq t \leq b$, it is clear that the function

$$g(s, t, n) = \sum_{r=1}^{n} \frac{\psi_r(s)\, \psi_r(t)}{\lambda_r - \lambda}$$

satisfies the requirements of the theorem of I., § 3, for these values of $s$ and $t$. We deduce that

$$\int_a^b K_\lambda(s, t) f(t)\, dt = \sum_{n=1} \frac{\psi_n(s) \int_a^b \psi_n(t) f(t)\, dt}{\lambda_n - \lambda}, \quad \ldots \ldots \quad (2)$$

and that the series on the right converges uniformly for values of $s$ in $(a, b)$.

It is easy to show directly that the series on the right of (2) is absolutely convergent. The result, however, follows at once from the fact that we have throughout left the order of the terms of the series arbitrary, which would otherwise have been impossible, by RIEMANN's theorem on derangement.

§ 2. We may here digress to prove a slight extension of the Hilbert-Schmidt expansion theorem,* applicable when $\kappa(s, t)$ has the properties mentioned above. Writing $\lambda = 0$ in (2) we have

$$\int_a^b \kappa(s, t) f(t)\, dt = \sum_{n=1} \frac{1}{\lambda_n} \psi_n(s) \int_a^b \psi_n(t) f(t)\, dt. \quad \ldots \ldots \quad (3)$$

Now, the function $\kappa(s, t)\, \psi_n(s) f(t)$ has a Lebesgue integral in the square $a \leq s \leq b$, $a \leq t \leq b$; further, the repeated integrals

$$\int_a^b \psi_n(s)\, ds \int_a^b \kappa(s, t) f(t)\, dt \qquad \int_a^b f(t)\, dt \int_a^b \kappa(s, t)\, \psi_n(s)\, ds$$

have a meaning. It follows from a known theorem† that the latter are equal; and hence that, as

$$\psi_n(t) = \lambda_n \int_a^b \kappa(s, t)\, \psi_n(s)\, ds,$$

$$\int_a^b g(t)\, \psi_n(t)\, dt = \frac{1}{\lambda_n} \int_a^b \psi_n(t) f(t)\, dt, \qquad \text{where} \qquad g(s) = \int_a^b \kappa(s, t) f(t)\, dt.$$

Supplying in (3) we see that

$$g(s) = \sum_{n=1} \psi_n(s) \int_a^b \psi_n(t)\, g(t)\, dt,$$

* HILBERT, 'Gött. Nachr.,' 1904, pp. 73–75. SCHMIDT, 'Math. Ann.,' vol. 63, pp. 451–2.
† HOBSON, 'The Theory of Functions of a Real Variable' (Cambridge, 1907), p. 582.

which is the result referred to. The series on the right is, of course, both absolutely and uniformly convergent.

It will be recalled that a direct application of SCHMIDT'S method of proof imposes narrower restrictions upon $f(s)$; it breaks down, for instance, if $[f(s)]^2$ is not integrable in $(a, b)$.

§ 3. Returning to the formula (2), it is evident that

$$-\lambda\int_a^b \mathrm{K}_\lambda(s,t)\,f(t)\,dt = \sum_{n=1} \frac{-\lambda}{\lambda_n-\lambda}\,\psi_n(s)\int_a^b \psi_n(t)\,f(t)\,dt. \quad . \quad . \quad . \quad . \quad (4)$$

This relation is true for all values of $\lambda$, other than the singular values of $\kappa(s,t)$, but in what follows it will not be necessary to consider values of $\lambda$ which are positive or zero. *As the assumption that $\lambda$ is always negative will make our work somewhat simpler we shall adopt it throughout this section.*

We proceed to investigate the behaviour of the right-hand member of this equality as $\lambda$ tends to $-\infty$. For this purpose we shall suppose that the order of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$, is that of non-decreasing magnitude. Thus far this order has not been material, but it will appear that the result obtained below turns upon the hypothesis stated.

Let $\sigma_n(s)$ be the sum of the first $n$ terms of the series

$$\psi_1(s)\int_a^b \psi_1(t)\,f(t)\,dt+\psi_2(s)\int_a^b \psi_2(t)\,f(t)\,dt+\ldots+\psi_n(s)\int_a^b \psi_n(t)\,f(t)\,dt+\ldots. \quad . \quad (5)$$

The sum of the first $m$ terms of the series on the right-hand side of (4) is

$$\frac{-\lambda}{\lambda_1-\lambda}\,\sigma_1(s)+\sum_{n=2}^{m}\frac{-\lambda}{\lambda_n-\lambda}\,[\sigma_n(s)-\sigma_{n-1}(s)],^*$$

which is

$$\sum_{n=1}^{m-1}\frac{-\lambda\,(\lambda_{n+1}-\lambda_n)}{(\lambda_{n+1}-\lambda)\,(\lambda_n-\lambda)}\,\sigma_n(s)+\frac{-\lambda}{\lambda_m-\lambda}\,\sigma_m(s). \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now, suppose that we can choose a positive integer $\mathrm{N}_1$ such that

$$\sigma_n(s) < h,$$

for all values of $n \geq \mathrm{N}_1$. If $m > \mathrm{N}_1$, the function of $\lambda$ (6) may be written

$$\sum_{n=1}^{\mathrm{N}_1-1}\frac{-\lambda\,(\lambda_{n+1}-\lambda_n)}{(\lambda_{n+1}-\lambda)\,(\lambda_n-\lambda)}\,\sigma_n(s)+\left[\sum_{n=\mathrm{N}_1}^{m-1}\frac{-\lambda\,(\lambda_{n+1}-\lambda_n)}{(\lambda_{n+1}-\lambda)\,(\lambda_n-\lambda)}\,\sigma_n(s)+\frac{-\lambda}{\lambda_m-\lambda}\,\sigma_m(s)\right].$$

Since the coefficients of the numbers $\sigma_n(s)$ are all positive when $\lambda$ is negative, we thus see that (6) is less than

$$\sum_{n=1}^{\mathrm{N}_1-1}\frac{-\lambda\,(\lambda_{n+1}-\lambda_n)}{(\lambda_{n+1}-\lambda)\,(\lambda_n-\lambda)}\,\sigma_n(s)+h\left[\sum_{n=\mathrm{N}_1}^{m-1}\frac{-\lambda\,(\lambda_{n+1}-\lambda_n)}{(\lambda_{n+1}-\lambda)\,(\lambda_n-\lambda)}+\frac{-\lambda}{\lambda_m-\lambda}\right];$$

* We here employ ABEL'S classical transformation. For the method of this paragraph, *cf.* BROMWICH, 'Proc. Lond. Math. Soc.' (1908), p. 59.

that is to say, less than

$$\sum_{n=1}^{N_1-1} \frac{-\lambda(\lambda_{n+1}-\lambda_n)}{(\lambda_{n+1}-\lambda)(\lambda_n-\lambda)} \sigma_n(s) + \frac{-\lambda}{\lambda_{N_1}-\lambda} h.$$

When any positive number $\epsilon$ is assigned, we can clearly choose a negative number $\Lambda$, whose numerical value is so great that the first term is less than $\epsilon$ for values of $\lambda \leq \Lambda$, and the second term is clearly less than $h$, for all values of $\lambda$. Observing that our choice of $\Lambda$ is quite independent of $m$, we deduce the inequality

$$\sum_{n=1} \frac{-\lambda}{\lambda_n-\lambda} \psi_n(s) \int_a^b \psi_n(t) f(t)\, dt \leq h+\epsilon, \qquad (\lambda \leq \Lambda_1),$$

which, in virtue of (4), may be written

$$-\lambda \int_a^b \mathrm{K}_\lambda(s,t) f(t)\, dt \leq h+\epsilon, \qquad (\lambda \leq \Lambda_1).$$

In a similar way, it may be shown that, if

$$\sigma_n(s) > k$$

for all values of $n \geq \mathrm{N}_2$, there exists a negative number $\Lambda_2$ such that

$$-\lambda \int_a^b \mathrm{K}_\lambda(s,t) f(t)\, dt \geq k+\epsilon,$$

for all values of $\lambda \leq \Lambda_2$.

§ 4. Let $\mathrm{U}(s)$ and $\mathrm{L}(s)$ be the upper and lower limits of indeterminacy of the series (5), it being supposed that, if necessary, these may have either of the improper values $\pm\infty$. In the first place, let us assume that $\mathrm{U}(s)$ is a finite number. Corresponding to any positive number $\epsilon$, we can then find a positive integer $\mathrm{N}_1$ such that

$$\sigma_n(s) < \mathrm{U}(s)+\epsilon$$

for all values of $n \geq \mathrm{N}_1$. It follows from what was said in the preceding paragraph that we can choose a negative number $\Lambda_1$ in such a way that

$$-\lambda \int_a^b \mathrm{K}_\lambda(s,t) f(t)\, dt \leq \mathrm{U}(s)+2\epsilon, \qquad (\lambda \leq \Lambda_1).$$

Thus we have

$$\overline{\lim_{\lambda \to -\infty}} -\lambda \int_a^b \mathrm{K}_\lambda(s,t) f(t)\, dt \leq \mathrm{U}(s). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Again, if $\mathrm{U}(s) = +\infty$, this inequality is obviously true, provided that we interpret "$x < \infty$" as "$x$ is a finite number, or has the improper value $-\infty$." Further, if $\mathrm{U}(s) = -\infty$, it is easily shown, by an argument similar to that just employed, that

$$\lim_{\lambda \to -\infty} -\lambda \int_a^b \mathrm{K}_\lambda(s,t) f(t)\, dt = -\infty.$$

It appears, therefore, that with the convention we have explained (7) is true in all cases; and in the same way it may be shown that

$$\varliminf_{\lambda \to -\infty} -\lambda \int_a^b K_\lambda(s, t) f(t)\, dt \geq L(s),$$

with a corresponding convention as to the meaning of $x > -\infty$. Adopting the hypotheses in § 1, we may sum up these results in the theorem :—

*If* $\psi_1(s), \psi_2(s), \ldots, \psi_n(s), \ldots$ *are a complete system of normal functions of* $\kappa(s, t)$, *arranged in such a way that the corresponding singular values are in non-decreasing order of magnitude, and if* $U(s)$, $L(s)$ *are respectively the upper and lower limits of indeterminacy of the series*

$$\psi_1(s) \int_a^b \psi_1(t) f(t)\, dt + \psi_2(s) \int_a^b \psi_2(t) f(t)\, dt + \ldots + \psi_n(s) \int_a^b \psi_n(t) f(t)\, dt + \ldots, \quad . \quad (5)$$

*then*

$$U(s) \geq \varlimsup_{\lambda \to -\infty} -\lambda \int_a^b K_\lambda(s, t) f(t)\, dt \geq \varliminf_{\lambda \to -\infty} -\lambda \int_a^b K_\lambda(s, t) f(t)\, dt \geq L(s).$$

In particular it is clear that, if the series (5) converges, its sum is

$$\lim_{\lambda \to -\infty} -\lambda \int_a^b K_\lambda(s, t) f(t)\, dt\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

while, if the series is non-oscillatory and divergent, (8) is $+\infty$, or $-\infty$, according as the series diverges to $+\infty$, or to $-\infty$.

§ 5. Let us now suppose that $\psi_1(s), \psi_2(s), \ldots, \psi_n(s), \ldots$, is a set of functions which are continuous in the interval $(a, b)$, and such that

$$\int_a^b \psi_n(s)\, \psi_m(s)\, ds = 0, \qquad (n \neq m),$$

$$= 1, \qquad (n = m).$$

For brevity, we shall refer to the set as *a system of normal functions for* $(a, b)$. We proceed to obtain two theorems which have important applications in the sequel.

Let $f(s)$ be any function whose square has a Lebesgue integral in $(a, b)$. The functions

$$f(s) - \sum_{n=1}^{m} \psi_n(s) \int_a^b \psi_n(t) f(t)\, dt, \qquad (m = 1, 2, \ldots),$$

have then the same property in virtue of the continuity of the functions $\psi_n(s)$. We deduce at once that, for all values of $m$,

$$\int_a^b \left[ f(s) - \sum_{n=1}^{m} \psi_n(s) \int_a^b \psi_n(t) f(t)\, dt \right]^2 ds = \int_a^b [f(s)]^2\, ds - \sum_{n=1}^{m} \left[ \int_a^b \psi_n(t) f(t)\, dt \right]^2.$$

As the left-hand member is not negative, it follows that, for all values of $m$,

$$\sum_{n=1}^{m}\left[\int_a^b \psi_n(t)\,f(t)\,dt\right]^2 \leq \int_a^b [f(t)]^2\,dt\,;*$$

and hence that *the series*

$$\left[\int_a^b \psi_1(t)\,f(t)\,dt\right]^2+\left[\int_a^b \psi_2(t)\,f(t)\,dt\right]^2+\ldots+\left[\int_a^b \psi_n(t)\,f(t)\,dt\right]^2+\ldots$$

*is convergent.*

§ 6. Since the $n^{\text{th}}$ term of a convergent series tends to zero, as $n$ increases indefinitely, it follows from the result obtained in the preceding paragraph that

$$\lim_{n\to\infty}\int_a^b \psi_n(t)\,f(t)\,dt = 0. \qquad (9)$$

Recalling the hypothesis in regard to $f(s)$, it is clear that this relation is true for all limited functions which have a Lebesgue integral in $(a, b)$. When it is possible to find a number $\overline{\psi}$ which, for all values of $n$, is greater than the upper limit of $|\psi_n(s)|$ in $(a, b)$, we may show that (9) is also true for all unlimited functions which possess a Lebesgue integral in $(a, b)$. For, assuming that $f(s)$ is unlimited, let us select a positive number N, and define a function $f_1(s)$ in $(a, b)$ by the rule

$$\begin{aligned} f_1(s) &= f(s) \qquad |f(s)| \leq \mathrm{N}, \\ &= 0 \qquad\quad |f(s)| > \mathrm{N}. \end{aligned}$$

If $\epsilon$ is any assigned positive number, it is known that N may be chosen great enough to ensure that

$$\int_a^b |f(t)-f_1(t)|\,dt < \epsilon.$$

Hence, with this choice of N, we have

$$\left|\int_a^b \psi_n(t)\,f(t)\,dt - \int_a^b \psi_n(t)\,f_1(t)\,dt\right| < \epsilon\overline{\psi}.$$

Again, since $f_1(s)$ is limited, we can find a positive integer such that, for this and all greater values of $n$, the numerical value of

$$\int_a^b \psi_n(t)\,f_1(t)\,dt$$

is less than $\epsilon$. For these values of $n$ we therefore have

$$\left|\int_a^b \psi_n(t)\,f(t)\,dt\right| < \epsilon\,(\overline{\psi}+1).$$

* This is sometimes called BESSEL'S inequality, *vide* the footnote on p. 56 of BÔCHER'S tract, 'An Introduction to the Study of Integral Equations' (1909).

We have thus established the theorem :—

*If the system of normal functions* $\psi_1(s), \psi_2(s), \ldots, \psi_n(s), \ldots$ *is such that the upper limit of* $|\psi_n(s)|$ *in* $(a, b)$ *is less than a fixed number, for all positive integral values of* $n$, *then*

$$\lim_{n\to\infty}\int_a^b \psi_n(t)\, f(t)\, dt = 0,$$

*provided that* $f(s)$ *has a Lebesgue integral in* $(a, b)$.

It will be seen at once that particular systems of normal functions for the interval $(0, \pi)$ are defined by

$$\text{(i)} \quad \psi_n(s) = \sqrt{\frac{2}{\pi}}\cos(n-1)s \qquad (n > 1),$$

$$= \sqrt{\frac{1}{\pi}} \qquad (n = 1),$$

$$\text{(ii)} \quad \psi_n(s) = \sqrt{\frac{2}{\pi}}\sin(n-\tfrac{1}{2})s \qquad (n \geq 1),$$

$$\text{(iii)} \quad \psi_n(s) = \sqrt{\frac{2}{\pi}}\cos(n-\tfrac{1}{2})s \qquad (n \geq 1),$$

$$\text{(iv)} \quad \psi_n(s) = \sqrt{\frac{2}{\pi}}\sin ns \qquad (n \geq 1).*$$

As each system satisfies the requirement of the theorem just stated, we see that :—

*If* $f(s)$ *is a function which has a Lebesgue integral in* $(0, \pi)$, *then*

$$\int_0^\pi f(t) \begin{matrix}\sin\\ \cos\end{matrix} \tfrac{1}{2}nt\, dt$$

*tends to the limit zero as the positive integer* $n$ *increases indefinitely.*

By applying this to the function which is equal to $f(s)$ in an interval $(\gamma, \delta)$ of $(0, \pi)$, and is zero elsewhere, we see that

$$\lim_{n\to\infty}\int_\gamma^\delta f(t) \begin{matrix}\sin\\ \cos\end{matrix} \tfrac{1}{2}nt\, dt = 0,$$

provided that $f(s)$ has a Lebesgue integral in $(\gamma, \delta)$.†

§ 7. Using the notation of §§ 3, 4, let us suppose that the normal functions of

* *Cf.* IV., §§ 9, 11, 12.

† The limitations that $(\gamma, \delta)$ should be within $(0, \pi)$, and that $n$ should be integral may be removed without difficulty, but the theorem enunciated is sufficient for our purposes. An alternative proof of the theorem in its most general form will be found in HOBSON'S 'Theory of Functions of a Real Variable,' (1907), pp. 674–5.

$\kappa(s, t)$ satisfy the requirement of the theorem of the preceding paragraph. Then, for any fixed value of $s$ belonging to $(a, b)$, we have

$$\lim_{n \to \infty} \psi_n(s) \int_a^b \psi_n(t) f(t)\, dt = 0.$$

It is easily proved from this that the numbers

$$\sigma_1(s),\ \sigma_2(s),\ \ldots,\ \sigma_n(s), \ldots$$

are everywhere dense in the interval $(\mathrm{L}(s), \mathrm{U}(s))$.*

We can therefore find a sequence of these numbers which has any given number belonging to the closed interval $(\mathrm{L}(s), \mathrm{U}(s))$ for its limit. Referring to the theorem of § 4, we see that :—

*If, in addition to the hypotheses of the theorem of § 4, it is assumed that the upper limit of* $|\psi_n(s)|$ *in* $(a, b)$ *is less than a fixed number for all values of n, then, by the introduction of suitable brackets, the series*

$$\psi_1(s) \int_a^b \psi_1(t) f(t)\, dt + \psi_2(s) \int_a^b \psi_2(t) f(t)\, dt + \ldots + \psi_n(s) \int_a^b \psi_n(t) f(t)\, dt \ldots$$

*may be made to converge to either of the limits*

$$\overline{\underline{\lim}}_{\lambda \to -\infty} -\lambda \int_a^b \mathrm{K}_\lambda(s, t) f(t)\, dt,$$

*provided that this limit is finite.*

§ 8. Returning to the system of normal functions of § 5, let us suppose it to be such that

$$\sum_{n=1} \left[\int_a^b \psi_n(t)\, \phi(t)\, dt\right]^2 = \int_a^b [\phi(t)]^2\, dt,$$

for each limited function $\phi(s)$ which has a Lebesgue integral in $(a, b)$. By considering the function which is identical with $\phi(s)$ in an interval $(a_1, b_1)$ of $(a, b)$ and is zero elsewhere, we see that

$$\sum_{n=1} \left[\int_{a_1}^{b_1} \psi_n(t)\, \phi(t)\, dt\right]^2 = \int_{a_1}^{b_1} [\phi(t)]^2\, dt,$$

provided that $\phi(s)$ is limited and has a Lebesgue integral in $(a_1, b_1)$.

Let us now suppose $f(s)$ to be a function, defined for all values of $s$, which has a Lebesgue integral in any finite interval. Let $\chi(s, t)$ be a limited function which has a Lebesgue integral with respect to $t$ in $(a_1, b_1)$, for each value of $s$ in a certain closed interval $(a_2, b_2)$; further, let us suppose that $\chi(s, t)$ is a uniformly continuous function of $s$ in $(a_2, b_2)$, for values of $t$ belonging to $(a_1, b_1)$. In virtue of the latter condition,

* *Cf.* HOBSON, 'The Theory of Functions of a Real Variable,' p. 712.

when any positive number $\epsilon$ is assigned, we can choose a positive number $\bar{\eta}$ small enough to ensure that

$$|\chi(s+\eta, t)-\chi(s, t)| < \epsilon,$$

for all values of $t$ in $(a_1, b_1)$ and for all pairs of points $s+\eta$, $s$ belonging to $(a_2, b_2)$ for which $|\eta| < \bar{\eta}$. Taking any fixed value of $s$, it follows from this that

$$\int_{a_1}^{b_1} |\chi(s+\eta, t)-\chi(s, t)|\, dt < \epsilon(b_1-a_1);$$

and hence that

$$\lim_{\eta \to 0} \int_{a_1}^{b_1} |\chi(s+\eta, t)-\chi(s, t)|\, dt = 0$$

at each point $s$ of $(a_2, b_2)$.

In the first place, let us suppose that $f(s)$ is a limited function; then

$$\chi(s, t) f(t+s)$$

is a limited function of $t$ and has a Lebesgue integral in $(a_1, b_1)$ for each value of $s$ in $(a_2, b_2)$. We therefore have

$$\sum_{n=1} \left[\int_{a_1}^{b_1} \psi_n(t)\, \chi(s, t) f(t+s)\, dt\right]^2 = \int_{a_1}^{b_1} [\chi(s, t) f(t+s)]^2\, dt, \quad . \quad . \quad . \quad (10)$$

for $a_2 \leq s \leq b_2$. The right-hand member of this equation and each of the terms of the series on the left may be shown to be continuous functions of $s$ in the interval $(a_2, b_2)$. For, supposing as above that $s$ and $s+\eta$ both belong to $(a_2, b_2)$, we have

$$\left|\int_{a_1}^{b_1} [\chi(s+\eta, t) f(t+s+\eta)]^2\, dt - \int_{a_1}^{b_1} [\chi(s, t) f(t+s)]^2\, dt\right|$$

$$\leq \left|\int_{a_1}^{b_1} \{[\chi(s+\eta, t)]^2-[\chi(s, t)]^2\}[f(t+s+\eta)]^2\, dt\right|$$

$$+\left|\int_{a_1}^{b_1} [\chi(s, t)]^2\{[f(t+s+\eta)]^2-[f(t+s)]^2\}\, dt\right|. \quad . \quad . \quad . \quad (11)$$

The first term on the right is not greater than

$$\bar{f}^2 \int_{a_1}^{b_1} |[\chi(s+\eta, t)]^2-[\chi(s, t)]^2|\, dt,$$

where $\bar{f}$ is the upper limit of $|f(s)|$; and the integral just written is not greater than

$$2\bar{\chi} \int_{a_1}^{b_1} |\chi(s+\eta, t)-\chi(s, t)|\, dt,$$

where $\bar{\chi}$ is the upper limit of $\chi(s, t)$ in the rectangle $a_1 \leq t \leq b_1$, $a_2 \leq s \leq b_2$. It follows from the remark made above that

$$\lim_{\eta \to 0} \int_{a_1}^{b_1} \{[\chi(s+\eta, t)]^2 - [\chi(s, t)]^2\} [f(t+s+\eta)]^2 \, dt = 0 \quad . \quad . \quad . \quad . \quad (12)$$

at each point $s$ of $(a_2, b_2)$.

Again, the second term on the right of (11) is not greater than

$$\bar{\chi}^2 \int_{a_1}^{b_1} |[f(t+s+\eta)]^2 - [f(t+s)]^2| \, dt;$$

and the integral just written tends to zero with $\eta$, by a theorem due to LEBESGUE.* Thus we have

$$\lim_{\eta \to 0} \int_{a_1}^{b_1} [\chi(s, t)]^2 \{[f(t+s+\eta)]^2 - [f(t+s)]^2\} \, dt = 0$$

at each point $s$ of $(a_2, b_2)$. Taking this in conjunction with (12), we see from (11) that

$$\int_{a_1}^{b_1} [\chi(s, t) f(t+s)]^2 \, dt$$

is a continuous function of $s$ in $(a_2, b_2)$. It may be proved in the same way that each of the terms

$$\int_{a_1}^{b_1} \psi_n(t) \chi(s, t) f(t+s) \, dt$$

is continuous in this interval.

It has now been shown that the positive series on the left of (10) has terms which are continuous in $(a_2, b_2)$, and a sum function which is also continuous in this interval. It follows from DINI'S theorem† that the series is uniformly convergent in $(a_2, b_2)$; and hence that, as $n$ tends to $\infty$,

$$\int_{a_1}^{b_1} \psi_n(t) \chi(s, t) f(t+s) \, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

converges to zero uniformly for $a_2 \leq s \leq b_2$.

§ 9. Let us next suppose $f(s)$ to be an unlimited function. We may show that (13) converges uniformly to zero, provided that the normal functions $\psi_n(s)$ satisfy the requirement of § 6. For, let us define a limited function $f_1(s)$ by the rule

$$\begin{aligned} f_1(s) &= f(s) \qquad & |f(s)| \leq \mathrm{N}, \\ &= 0 \qquad & |f(s)| > \mathrm{N}; \end{aligned}$$

* *Vide* 'Leçons sur les Séries Trigonométriques' (1906), pp. 15, 16.

† DINI, 'Fondamente per la teoria delle funzioni di variabili reali' (Pisa, 1878), § 99. See also YOUNG, "On Monotone Sequences of Continuous Functions," 'Proc. Camb. Phil. Soc.,' vol. xiv., pp. 520–523. A proof of the theorem will be found in HOBSON'S 'Theory of Functions of a Real Variable,' pp. 478–479.

and, assuming an arbitrarily small positive, $\epsilon$, to be assigned, let N be chosen great enough to ensure that

$$\int_{a_1+a_2}^{b_1+b_2} |f(t)-f_1(t)|\, dt < \epsilon.$$

Then we have

$$\left|\int_{a_1}^{b_1} \psi_n(t)\chi(s,t)\, f(t+s)\, dt - \int_{a_1}^{b_1} \psi_n(t)\chi(s,t)\, f_1(t+s)\, dt\right| \leq \overline{\psi}\,\overline{\chi}\int_{a_1}^{b_1} |f(t+s)-f_1(t+s)|\, dt,$$

which is certainly less than $\epsilon\overline{\psi}\,\overline{\chi}$, for all values of $s$ in $(a_2, b_2)$. Again, since $f_1(t+s)$ is limited, we can choose a positive integer $m$ such that

$$\left|\int_{a_1}^{b_1} \psi_n(t)\chi(s,t)\, f_1(t+s)\, dt\right| < \epsilon,$$

for $n \geq m$ and $a_2 \leq s \leq b_2$. It follows that, for these values of $n$ and $s$, we have

$$\left|\int_{a_1}^{b_1} \psi_n(t)\chi(s,t)\, f(t+s)\, dt\right| < \epsilon(\overline{\psi}\,\overline{\chi}+1);$$

and hence that

$$\int_{a_1}^{b_1} \psi_n(t)\chi(s,t)\, f(t+s)\, dt$$

converges uniformly to zero in $(a_2, b_2)$.

It may be proved in the same way that each of the other integrals

$$\int_{a_1}^{b_1} \psi_n(t)\chi(s,t)\, f(\pm t\pm s)\, dt$$

has this property. Hence the theorem :—

*Let the system of normal functions* $\psi_1(s), \psi_2(s), \ldots, \psi_n(s), \ldots$ *be such that* (1) *the upper limit of* $|\psi_n(s)|$ *in* $(a, b)$ *is less than a fixed number, for all positive integral values of* $n$, *and* (2)

$$\sum_{n=1}\left[\int_a^b \psi_n(t)\,\phi(t)\, dt\right]^2 = \int_a^b [\phi(t)]^2\, dt,$$

*for each limited function* $\phi(s)$ *which has a Lebesgue integral in* $(a, b)$. *Let* $f(s)$ *be a function, defined for all values of* $s$, *which has a Lebesgue integral in any finite interval. Let* $\chi(s, t)$ *be a limited function which, for each value of* $s$ *in an interval* $(a_2, b_2)$, *has a Lebesgue integral with respect to* $t$ *in an interval* $(a_1, b_1)(a \leq a_1 < b_1 \leq b)$; *and let* $\chi(s, t)$ *be a uniformly continuous function of* $s$ *in* $(a_2, b_2)$ *for values of* $t$ *belonging to* $(a_1, b_1)$. *Then, as* $n$ *tends to* $\infty$, *each of the four integrals*

$$\int_{a_1}^{b_1} \psi_n(t)\chi(s,t)\, f(\pm t\pm s)\, dt$$

*converges uniformly to zero in the interval* $(a_2, b_2)$.

The theorem we have just enunciated is of very general character, and may be stated in a variety of particular forms. Without exhausting all possibilities, we shall mention two corollaries which will be of use in the sequel.

It has already been pointed out that the four systems of normal functions defined in § 6 satisfy the condition (1) of the above theorem; and it will be shown, in a paper to be published shortly,* that the condition (2) is also satisfied. Further, a function $\chi(t)$, which has a Lebesgue integral in an interval $(\gamma, \delta)$ $(0 \leq \gamma < \delta \leq \pi)$, may be regarded as a uniformly continuous function of $s$ in any interval whatever, for values of $t$ belonging to $(\gamma, \delta)$. We have thus the first of the corollaries mentioned:—

*Let $f(s)$ be a function, defined for all values of $s$, which has a Lebesgue integral in any finite interval; and let $\chi(t)$ be a limited function which has a Lebesgue integral in an interval $(\gamma, \delta)$ $(0 \leq \gamma < \delta \leq \pi)$. Then, as the positive integer $n$ increases indefinitely, each of the eight integrals*

$$\int_\gamma^\delta f(\pm s \pm t)\,\chi(t)\,{\sin \atop \cos}\,\tfrac{1}{2}nt\,dt$$

*converges uniformly to zero in any finite interval.*†

A sufficient condition that $\chi(s, t)$ may be a uniformly continuous function of $s$ in $(\gamma', \delta')$, for values of $t$ belonging to $(\gamma, \delta)$ is that $\chi(s, t)$ should be a continuous function of the two variables $s$ and $t$ in the rectangle $\gamma' \leq s \leq \delta'$, $\gamma \leq t \leq \delta$. Hence we have the second corollary:—

*Let $f(s)$ be a function, defined for all values of $s$, which has a Lebesgue integral in any finite interval; and let $\chi(s, t)$ be a continuous function of the two variables*

* In this paper I shall prove the following theorem:—

*Let $f(s)$ be any function whose square has a Lebesgue integral in $(0, \pi)$. Let $\psi_1(s)$, $\psi_2(s)$, …, $\psi_n(s)$, … be the complete system of normal functions which, for suitable values of $\mu$, satisfy the differential equation*

$$\frac{d^2u}{ds^2} + (q+\mu)\,u = 0,$$

*and an assigned pair of boundary conditions at the end points of $(0, \pi)$. Then the series*

$$\left[\int_0^\pi \psi_1(t) f(t)\,dt\right]^2 + \left[\int_0^\pi \psi_2(t) f(t)\,dt\right]^2 + \ldots + \left[\int_0^\pi \psi_n(t) f(t)\,dt\right]^2 + \ldots$$

*is convergent and its sum is*

$$\int_0^\pi [f(t)]^2\,dt$$

*in all cases.*

We shall see below (IV., §§ 9, 11, 12) that the four systems of normal functions each satisfy the requirements of this theorem.

In the meantime the reader will be able to deduce the particular cases here required from a result obtained by A. C. DIXON ("On a Property of Summable Functions," 'Proc. Camb. Phil. Soc.,' vol. xv., pp. 211–216).

† The limitation that $n$ should be integral may be removed without difficulty. HOBSON has proved the equivalent of this corollary for the case in which $\chi(t)$ has limited total fluctuation in $(\gamma, \delta)$, *vide* "On the Uniform Convergence of FOURIER'S Series," 'Proc. Lond. Math. Soc.,' ser. 2, vol. 5, pp. 277–281; 'The Theory of Functions of a Real Variable' (1907), pp. 683–687.

*s and t in the rectangle* $\gamma' \leq s \leq \delta'$, $\gamma \leq t \leq \delta$ $(0 \leq \gamma < \delta \leq \pi)$. *Then, as the positive integer n increases indefinitely, each of the eight integrals*

$$\int_{\gamma}^{\delta} f(\pm s \pm t)\, \chi(s, t) \begin{matrix}\sin \\ \cos\end{matrix} \tfrac{1}{2}nt\, dt$$

*converges uniformly to zero in* $(\gamma', \delta')$.

§ 10. From the first corollary of the preceding paragraph, it is possible to deduce a theorem which we shall have to employ below. This theorem depends upon the following lemma :—*

*Let* $g(s, n)$ *be a limited function defined for all values of s in an interval* $(a, b)$, *and for all positive integral values of n; also let this function converge to* $g(s)$, *as n increases indefinitely, uniformly in* $(a, b)$. *Then, as n increases indefinitely, the arithmetic mean*

$$\frac{1}{n}\{g(s, 1) + g(s, 2) + \ldots + g(s, n)\}$$

*converges uniformly to* $g(s)$ *in* $(a, b)$.

To prove this, we observe that, when any positive number $\epsilon$ is assigned, a positive integer $N_1$ may be chosen great enough to ensure that

$$|g(s, n) - g(s)| < \tfrac{1}{2}\epsilon$$

for all values of $n \geq N_1$, and of $s$ in $(a, b)$. It follows at once that we have

$$\left|\frac{g(s, 1) + g(s, 2) + \ldots + g(s, n)}{n} - g(s)\right| < \frac{1}{n}\left|\sum_{r=1}^{N_1 - 1}\{g(s, r) - g(s)\}\right| + \tfrac{1}{2}\epsilon.$$

As $g(s, n)$ is limited, it is clearly possible to choose a positive integer $N_2$ in such a way that the first term on the right is less than $\tfrac{1}{2}\epsilon$, for all values of $n \geq N_2$, and of $s$ in $(a, b)$. Hence we see that, when $n$ is not less than the greater of $N_1$ and $N_2$,

$$\left|\frac{1}{n}\{g(s, 1) + g(s, 2) + \ldots + g(s, n)\} - g(s)\right| < \epsilon,$$

for all values of $s$ in $(a, b)$. The lemma is therefore established.

With the notation of the previous paragraph, let us write

$$g(s, n) = \int_{\gamma}^{\delta} f(s + t)\, \chi(t) \sin \tfrac{1}{2}(2n - 1)\, t\, dt.$$

Clearly $g(s, n)$ satisfies the requirements of the above lemma, the function $g(s)$ being everywhere zero, and $(a, b)$ any finite interval whatever. Since

$$\sum_{r=1}^{n} \sin \tfrac{1}{2}(2r - 1)\, t = \sin^2 \tfrac{1}{2}nt / \sin \tfrac{1}{2}t,$$

* *Cf.* HARDY, 'Proc. Lond. Math. Soc.,' ser. 2, vol. 4, p. 257.

it follows from the lemma just established that

$$\frac{1}{n}\int_{\gamma}^{\delta} f(s+t)\,\chi(t)\,\frac{\sin^2 \frac{1}{2}nt}{\sin \frac{1}{2}t}\,dt$$

converges uniformly to zero in any finite interval.

The same remarks being applicable when $f(s+t)$ is replaced by either of the functions $f(s-t)$, $f(-s+t)$, $f(-s-t)$, we have the theorem :—

*With the hypotheses of the first corollary of* § 10, *each of the four functions*

$$\frac{1}{n}\int_{\gamma}^{\delta} f(\pm s\pm t)\,\chi(t)\,\frac{\sin^2 \frac{1}{2}nt}{\sin \frac{1}{2}t}\,dt$$

*converges uniformly to zero for values of s in any finite interval, as the positive integer n increases indefinitely.*

It will be clear that a variety of results may be obtained from the corollary referred to by a similar process.

## III.—A Method of Representing an Arbitrary Function in Terms of Solutions of a Sturm-Liouville Equation. General Theorems on the Behaviour of Sturm-Liouville Series.

§ 1. We proceed to apply the foregoing results to the theory of Sturm-Liouville series. With this in view, we shall commence by considering those solutions of the differential equation

$$\frac{d}{dx}\left(k\frac{dv}{dx}\right)+(gr-l)\,v=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which, by a suitable choice of the parameter $r$, can be made to satisfy a certain pair of boundary conditions at the ends of an interval $(a, b)$. In what follows it will be assumed that in the closed interval $(a, b)$ (1) $l$ is a continuous function of $x$, (2) $g$ and $k$ are continuous functions of $x$ which never vanish, (3) $k$ possesses a continuous differential coefficient, and (4) $gk$ has a continuous derivative of the second order.

The pair of boundary conditions above referred to will be supposed to be one of the following four :—

$$\text{(i)}\quad \left.\begin{aligned} &\frac{dv}{dx}-hv=0 \quad \text{at} \quad x=a \\ &\frac{dv}{dx}+\mathrm{H}v=0 \quad \text{,,} \quad x=b \end{aligned}\right\}$$

$$\text{(ii)}\quad \left.\begin{aligned} &v=0 \quad \text{,,} \quad x=a \\ &\frac{dv}{dx}+\mathrm{H}v=0 \quad \text{,,} \quad x=b \end{aligned}\right\}$$

$$\text{(iii)}\quad \left.\begin{array}{lll} \dfrac{dv}{dx}-hv=0 & \text{at} & x=a \\ v=0 & ,, & x=b \end{array}\right\}$$

$$\text{(iv)}\quad \left.\begin{array}{lll} v=0 & ,, & x=a \\ v=0 & ,, & x=b \end{array}\right\}$$

The constants $h$ and H will be supposed real, but will be otherwise unrestricted.

In the sequel we shall employ the symbols B, ${}_a\text{B}$, ${}^b\text{B}$, ${}^b_a\text{B}$ to denote the pairs of boundary conditions numbered (i), (ii), (iii), (iv), respectively. It will be observed that the letters which stand to the left of B indicate the end points at which the boundary condition is $v = 0$.

§ 2. With the hypotheses of the preceding paragraph, we may transform (1) by means of the substitution

$$s = \xi \int_a^x (g/k)^{1/2}\, dx, \qquad u = (gk)^{1/4}\, v, \qquad \mu = r\xi^{-2},$$

where $\xi$ is the constant

$$\frac{\pi}{\int_a^b (g/k)^{1/2}\, dx.}$$

The differential equation then becomes

$$\frac{d^2u}{ds^2} + (q+\mu)\, u = 0, \qquad (2)$$

where

$$q = -\left(\frac{w''(s)}{w(s)} + \frac{j(s)}{\xi^2}\right),$$

$w(s)$ and $j(s)$ being the functions $(gk)^{1/4}$ and $\dfrac{l}{g}$ respectively, expressed as functions of $s$. In virtue of our hypotheses, it will be clear that $q$ is a continuous function of $s$.

Corresponding to the interval $a \leq x \leq b$ we have the interval $0 \leq s \leq \pi$, and to each pair of boundary conditions B, ${}_a\text{B}$, ${}^b\text{B}$, ${}^b_a\text{B}$ for the former interval there corresponds a pair for the latter; these we shall denote by B′, ${}_0\text{B}'$, ${}^\pi\text{B}'$, ${}^\pi_0\text{B}'$ respectively. The reader will find that the pair of boundary conditions B′ is

$$\left.\begin{array}{lll} \dfrac{du}{ds}-h'u=0 & \text{at} & s=0 \\ \dfrac{du}{ds}+\text{H}'u=0 & ,, & s=\pi \end{array}\right\},$$

where $h'$ and H′ are real constants whose values depend upon $h$ and H respectively; he will also find that the pair ${}^\pi_0\text{B}'$ is

$$\left.\begin{array}{lll} u=0 & \text{at} & s=0 \\ u=0 & ,, & s=\pi \end{array}\right\}.$$

The pairs of boundary conditions ${}_0\text{B}'$, ${}^\pi\text{B}'$ will then be obvious.

§ 3. In what follows we shall consider in detail only the case when the pair of boundary conditions for $(a, b)$ is B. A slight modification of the method developed below is necessary to obtain a formal proof when the pair of conditions is ${}_a\mathrm{B}$, ${}^b\mathrm{B}$, or ${}_a^b\mathrm{B}$, but the nature of this modification is so obvious that we shall content ourselves with a statement of the corresponding results.

After what has been said in the previous paragraph, it is clear that the problem of determining the solutions of (1) which satisfy B is equivalent to that of determining the solutions of (2) which, for suitable values of $\mu$, satisfy B′.

It may happen that certain of these values of $\mu$ are not all positive. If so, we can choose a number $\kappa$ which is less than the least of them. The equation

$$\frac{d^2u}{ds^2}+(q+\kappa+\mu)\,u=0$$

is then such that the values of $\mu$ for which there exist solutions satisfying B′ are all positive, and clearly the aggregate of these solutions is identical with the aggregate of the solutions of (2) which satisfy B′. It follows that, without loss of generality, we may suppose the values of $\mu$ for which there exist solutions of (2) satisfying B′ to be all positive.

It has been shown by KNESER that the GREEN'S function* of

$$\frac{d^2u}{ds^2}+qu=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

for the pair of boundary conditions B′ is

$$\begin{aligned}\kappa(s,t) &= \theta(s)\,\phi(t) \quad (s \leqq t)\\ &= \phi(s)\,\theta(t) \quad (s \geqq t),\end{aligned}$$

where $\theta(s)$† satisfies (3) and the boundary condition

$$\frac{du}{ds}-h'u=0 \quad \text{at} \quad s=0,$$

$\phi(s)$† satisfies (3) and the boundary condition

$$\frac{du}{ds}+\mathrm{H}'u=0 \quad \text{at} \quad s=\pi,$$

and the two functions are chosen in such a way that

$$\theta(s)\,\phi'(s)-\phi(s)\,\theta'(s)=-1.$$

* For the theory of GREEN'S functions, see HILBERT, 'Gött. Nachr.' (1904), pp. 214–234, and KNESER, 'Math. Ann.,' vol. 63, pp. 482–486.

† From a theorem on linear differential equations it is known that, as $q$ is continuous, these solutions exist and have continuous second derivations in the interval $(0, \pi)$ (*vide* PICARD, 'Traité d'Analyse,' tome III. (1896), p. 92).

The values of $\mu$, say $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$, for which there exist solutions of (2) satisfying B′ are known to be the roots of the determinant of $\kappa(s, t)$. Further, if $\psi_1(s)$, $\psi_2(s), \ldots, \psi_n(s), \ldots$ respectively are these solutions, chosen in such a way that

$$\int_0^\pi [\psi_n(s)]^2\, ds = 1 \quad (n = 1, 2, \ldots),$$

it is known that they are the complete system of normal functions of $\kappa(s,t)$. Recalling that $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ are all positive, it follows from the theorem quoted above* that

$$\kappa(s, t) = \sum_{n=1} \frac{\psi_n(s)\, \psi_n(t)}{\lambda_n} \quad \ldots \ldots \quad (4)$$

§ 4. Consider now the effect of replacing $q$ by $q+\lambda$ in (2), where $\lambda$ is a negative number.† The values of $\mu$ for which there exist solutions of

$$\frac{d^2u}{ds^2} + (q+\lambda+\mu)\, u = 0$$

satisfying B′ are clearly $\lambda_1-\lambda, \lambda_2-\lambda, \ldots, \lambda_n-\lambda, \ldots$, and the solutions corresponding to these are $\psi_1(s), \psi_2(s), \ldots, \psi_n(s), \ldots$, respectively. It follows from (4) that the GREEN'S function of

$$\frac{d^2u}{ds^2} + (q+\lambda)\, u = 0 \quad \ldots \ldots \quad (5)$$

is

$$\sum_{n=1} \frac{\psi_n(s)\, \psi_n(t)}{\lambda_n - \lambda},$$

*i.e.*, $K_\lambda(s, t)$, the solving function corresponding to $\kappa(s, t)$. But, by KNESER'S theorem, the GREEN'S function of (5) for the pair of boundary conditions B′ is the function defined by

$$\Theta_\lambda(s)\, \Phi_\lambda(t) \quad (s \leq t)$$
$$\Phi_\lambda(s)\, \Theta_\lambda(t) \quad (s \geq t),$$

where $\Theta_\lambda(s)$ satisfies the equation (5) and the boundary condition

$$\frac{du}{ds} - h'u = 0 \quad \text{at} \quad s = 0,$$

and $\Phi_\lambda(s)$ satisfies this equation and the boundary condition

$$\frac{du}{ds} + H'u = 0 \quad \text{at} \quad s = \pi,$$

* II., § 1.

† For our immediate purpose this restriction may be replaced by the wider one that $\lambda$ is not equal to one of the singular values $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$. As the condition that $\lambda$ should be negative is forced upon us in the following paragraph, and we shall not need to consider other values, we shall continue to suppose that $\lambda$ is negative (*vide* II., § 3).

the two functions being such that

$$\Theta_\lambda(s)\,\Phi'_\lambda(s)-\Phi_\lambda(s)\,\Theta'_\lambda(s) = -1.$$

Writing

$$\theta_\lambda(s) = \frac{\Theta_\lambda(s)}{\Theta_\lambda(0)}, \qquad \phi_\lambda(s) = \frac{\Phi_\lambda(s)}{\Phi_\lambda(\pi)},$$

we thus see that

$$\begin{aligned} K_\lambda(s, t) &= \frac{\theta_\lambda(s)\,\phi_\lambda(t)}{\delta(\lambda)} \quad (s \leq t) \\ &= \frac{\phi_\lambda(s)\,\theta_\lambda(t)}{\delta(\lambda)} \quad (s \geq t), \end{aligned} \qquad (6)$$

where $\theta_\lambda(s)$ is the solution of (5) defined by the conditions $\theta_\lambda(0) = 1$, $\theta'_\lambda(0) = h'$, $\phi_\lambda(s)$ is the solution defined by $\phi_\lambda(\pi) = 1$, $\phi'_\lambda(\pi) = -H'$, and $\delta(\lambda)$ is the value of

$$\phi_\lambda(s)\,\theta'_\lambda(s)-\theta_\lambda(s)\,\phi'_\lambda(s),$$

which is known to be independent of $s$.

§ 5. The result just stated may be employed to obtain an asymptotic formula for $K_\lambda(s, t)$, when $\lambda$ is negative and numerically large. For this purpose it will be convenient to write $\lambda = -\rho^2$, where $\rho$ is supposed real and positive. If we denote by D the operator $\frac{d}{ds}$, the equation (2) then becomes

$$[D^2-\rho^2]\,u = -qu.$$

The complete primitive of this is

$$\begin{aligned} u &= c_1 \cosh \rho s + c_2 \sinh \rho s - [D^2-\rho^2]^{-1}\,qu \\ &= c_1 \cosh \rho s + c_2 \sinh \rho s - \frac{1}{2\rho}\{[D-\rho]^{-1}\,qu-[D+\rho]^{-1}\,qu\} \\ &= c_1 \cosh \rho s + c_2 \sinh \rho s - \frac{1}{\rho}\int_0^s q_1 u_1 \sinh \rho\,(s-s_1)\,ds_1, \end{aligned}$$

where $c_1$ and $c_2$ are constants, and $q_1$, $u_1$ are what $q$, $u$ become when $s_1$ is substituted for $s$.

If $u = \theta_\lambda(s)$, the conditions $\theta_\lambda(0) = 1$, $\theta'_\lambda(0) = h'$ give

$$c_1 = 1, \quad c_2 = \frac{h'}{\rho}.$$

Accordingly we have

$$\theta_\lambda(s) = \cosh \rho s + \frac{h'}{\rho}\sinh \rho s - \frac{1}{\rho}\int_0^s q_1 \theta_\lambda(s_1) \sinh \rho\,(s-s_1)\,ds_1. \qquad (7)$$

If we write

$$\zeta_\lambda(s) = \frac{\theta_\lambda(s)}{\cosh \rho s}, \qquad (8)$$

this equation becomes

$$\zeta_\lambda(s) = 1 + \frac{h'}{\rho}\tanh \rho s - \frac{1}{\rho}\int_0^s q_1 \zeta_\lambda(s_1)\,\frac{\cosh \rho s_1 \sinh \rho\,(s-s_1)}{\cosh \rho s}\,ds_1. \qquad (9)$$

Now at all points of the closed interval $(0, \pi)$

$$0 \leq \tanh \rho s < 1\,;$$

and, whatever value belonging to the interval $s$ may have,

$$\left|\frac{\cosh \rho s_1 \,.\, \sinh \rho\,(s-s_1)}{\cosh \rho s}\right| = \tfrac{1}{2}\left|\frac{\sinh \rho s + \sinh \rho\,(s-2s_1)}{\cosh \rho s}\right| \leq \tfrac{1}{2}\tanh \rho s + \tfrac{1}{2}\left|\frac{\sinh \rho\,(s-2s_1)}{\cosh \rho s}\right| \leq 1,$$

if $s_1$ is a point of the closed interval $(0, s)$. Hence, if $\bar{\zeta}_\lambda$ be the greatest value of $|\zeta_\lambda(s)|$ in $(0, \pi)$, we see from (9) that

$$\bar{\zeta}_\lambda \leq 1 + \frac{|h'|}{\rho} + \frac{\bar{\zeta}_\lambda}{\rho}\int_0^\pi |q_1|\, ds_1,$$

which may be written

$$\bar{\zeta}_\lambda \leq \left(1 + \frac{|h'|}{\rho}\right)\left\{1 - \frac{1}{\rho}\int_0^\pi |q_1|\, ds_1\right\}^{-1}.$$

It follows that $\zeta_\lambda(s)$ is limited for values of $\rho$ that are greater than a certain positive number, and of $s$ that lie in the closed interval $(0, \pi)$.

§ 6. It will be convenient in what follows to use $\alpha(\rho, s)$ as a shorthand symbol for the phrase "a function of $\rho$ and $s$ which is limited for values of $\rho$ that are greater than a certain positive number, and of $s$ that lie in the closed interval $(0, \pi)$."

With this convention it follows from the result obtained in the preceding paragraph, that (9) may be written

$$\zeta_\lambda(s) = 1 + \frac{\alpha(\rho, s)}{\rho}, \qquad (10)$$

whence, in virtue of (8), we obtain the formula

$$\theta_\lambda(s) = \cosh \rho s\left(1 + \frac{\alpha(\rho, s)}{\rho}\right). \qquad (11)$$

In order to obtain an asymptotic formula for $\theta'_\lambda(s)$, we turn back to the equation (7). Differentiating with respect to $s$, we obtain

$$\theta'_\lambda(s) = \rho \sinh \rho s + h' \cosh \rho s - \int_0^s q_1 \theta_\lambda(s_1) \cosh \rho\,(s-s_1)\, ds_1.$$

Using (11) this becomes

$$\theta'_\lambda(s) = \rho \sinh \rho s + h' \cosh \rho s - \int_0^s q_1 \cosh \rho s_1 \cosh \rho\,(s-s_1)\, ds_1$$
$$- \frac{1}{\rho}\int_0^s q_1 \alpha(\rho, s_1) \cosh \rho\,(s-s_1)\, ds_1. \qquad (12)$$

Since

$$\frac{\cosh \rho s_1 \,.\, \cosh \rho\,(s-s_1)}{\cosh \rho s} \leq 1$$

for all values of $s_1$ which belong to the interval $(0, s)$, it appears that the third term on the right-hand side of (12) is of the form

$$\alpha(\rho, s) \cosh \rho s;$$

and it is evident that the fourth is of the same form. Thus (12) gives

$$\theta'_\lambda(s) = \rho \sinh \rho s + \alpha(\rho, s) \cosh \rho s.$$

Proceeding in a similar manner, we may obtain analogous formulæ for $\phi_\lambda(s)$ and $\phi'_\lambda(s)$. It is, however, more expeditious to deduce these from the formulæ already obtained for $\theta_\lambda(s)$ and $\theta'_\lambda(s)$ by a device which we proceed to explain. Putting in evidence the argument of $q$, let $u(s)$ be the solution of

$$\frac{d^2u}{ds^2} + [q(\pi - s) + \lambda]u = 0$$

in the interval $(0, \pi)$, which satisfies the conditions $u = 1$, $\frac{du}{ds} = \mathrm{H}'$, at $s = 0$. Clearly $u(\pi - s)$ is the solution of

$$\frac{d^2u}{ds^2} + [q(s) + \lambda]u = 0$$

in this interval which satisfies the conditions $u = 1, \frac{du}{ds} = -\mathrm{H}'$, at $s = \pi$. Recalling that the asymptotic formula (11) is valid for all values of $h'$, and for all continuous functions $q$, we deduce from it the formula

$$\phi_\lambda(s) = \cosh \rho(\pi - s)\left(1 + \frac{\alpha(\rho, s)}{\rho}\right);$$

and similarly, in virtue of the relation

$$\frac{d}{ds}[u(\pi - s)] = -u'(\pi - s),$$

we obtain

$$\phi'_\lambda(s) = -\rho \sinh \rho(\pi - s) - \alpha(\rho, s) \cosh \rho(\pi - s)$$

from the formula given for $\theta'_\lambda(s)$.

§ 7. Supplying the formulæ of the preceding paragraph in

$$\delta(\lambda) = \phi_\lambda(s)\,\theta'_\lambda(s) - \theta_\lambda(s)\,\phi'_\lambda(s),$$

we obtain

$$\delta(\lambda) = \cosh \rho(\pi - s)[\rho \sinh \rho s + \alpha(\rho, s) \cosh \rho s]\left(1 + \frac{\alpha(\rho, s)}{\rho}\right)$$
$$+ \cosh \rho s[\rho \sinh \rho(\pi - s) + \alpha(\rho, s) \cosh \rho(\pi - s)]\left(1 + \frac{\alpha(\rho, s)}{\rho}\right),$$

where it must be borne in mind that the various symbols $\alpha(\rho, s)$ do not necessarily refer to the same function On multiplying this out it appears that the terms which do not involve $\alpha(\rho, s)$ give $\rho \sinh \rho\pi$; and each of the remaining terms is easily shown to be of the form

$$\alpha(\rho, s) \sinh \rho\pi.$$

Recalling that $\delta(\lambda)$ is independent of $s$, we thus see that

$$\delta(\lambda) = \rho \sinh \rho\pi \left(1 + \frac{\alpha(\rho)}{\rho}\right),$$

where, in analogy with the notation explained in § 6, $\alpha(\rho)$ is a shorthand symbol for "a function of $\rho$ which is limited for values of the argument that are greater than a certain fixed positive number." If, in a similar way, we use $\alpha(\rho, s, t)$ to denote "a function of $\rho$, $s$ and $t$ which is limited for values of $\rho$ that are greater than a certain positive number, and of $s$ and $t$ that lie both in the closed interval $(0, \pi)$," the formulæ of this and the preceding paragraph, when supplied in the expressions for $K_\lambda(s, t)$ obtained in § 4, give

$$K_\lambda(s, t) = \Gamma_\lambda(s, t)\left(1 + \frac{\alpha(\rho, s, t)}{\rho}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$\Gamma_\lambda(s, t) = \frac{\cosh \rho s \cosh \rho(\pi - t)}{\rho \sinh \rho\pi} \quad (s \leq t)$$

$$= \frac{\cosh \rho(\pi - s) \cosh \rho t}{\rho \sinh \rho\pi} \quad (s \geq t).$$

§ 8. Let $f(s)$ be any function which has a Lebesgue integral in the interval $(0, \pi)$. Then from (13) we obtain

$$-\lambda \int_0^\pi [K_\lambda(s, t) - \Gamma_\lambda(s, t)] f(t)\, dt = \rho \int_0^\pi \Gamma_\lambda(s, t)\, \alpha(\rho, s, t) f(t)\, dt.$$

Now when $s \geq t$ we have

$$\rho\Gamma_\lambda(s, t) = \frac{\cosh \rho(\pi - s + t) + \cosh \rho(\pi - s - t)}{2 \sinh \rho\pi};$$

hence, recalling that $\Gamma_\lambda(s, t)$ is a symmetric function of $s$ and $t$, we see that

$$\rho\Gamma_\lambda(s, t) \leq \coth \rho\pi,$$

for $0 \leq s \leq \pi$, $0 \leq t \leq \pi$.

Since $\coth \rho\pi$, considered as a function of $\rho$, is limited in any range which does not include points within an arbitrarily small distance from the origin, it appears that, for $0 \leq s \leq \pi$, $0 \leq t \leq \pi$, and for all values of $\rho$ greater than any assigned positive number, $\rho\Gamma_\lambda(s, t)$ is limited. The same remark therefore applies to the function $\rho\Gamma_\lambda(s, t)\, \alpha(\rho, s, t)$.

Again we have

$$\lim_{\rho \to \infty} \rho\Gamma_\lambda(s, t) = 0 \quad (s \neq t) \qquad = 1 \quad (s = t).$$

It follows that, for unequal values of $s$ and $t$,

$$\lim_{\rho \to \infty} \rho \Gamma_\lambda (s, t)\, \alpha (\rho, s, t) = 0.$$

Applying the theorem of I, § 1, we deduce that, as $\rho$ tends to $\infty$,

$$\rho \int_0^\pi \Gamma_\lambda (s, t)\, \alpha (\rho, s, t) f(t)\, dt,$$

*i.e.*,

$$-\lambda \int_0^\pi [\mathrm{K}_\lambda (s, t) - \Gamma_\lambda (s, t)] f(t)\, dt$$

tends to zero, for all values of $s$ in $(0, \pi)$.

In what follows we shall express the result just obtained by the notation

$$-\lambda \int_0^\pi \mathrm{K}_\lambda (s, t) f(t)\, dt \underset{\lambda \to -\infty}{=\!=\!=} -\lambda \int_0^\pi \Gamma_\lambda (s, t) f(t)\, dt,$$

the symbol between the left- and right-hand members indicating that their difference tends to zero, as $\lambda$ tends to $-\infty$.

§ 9. Let us now consider

$$-\lambda \int_0^\pi \Gamma_\lambda (s, t) f(t)\, dt.$$

The value of this is evidently

$$\frac{\rho \cosh \rho (\pi - s)}{\sinh \rho\pi} \int_0^s \cosh \rho t\, f(t)\, dt + \frac{\rho \cosh \rho s}{\sinh \rho\pi} \int_s^\pi \cosh \rho (\pi - t) f(t)\, dt. \quad . \quad (14)$$

Now, whatever value $s$ may have in the closed interval $(0, \pi)$,

$$\left| \frac{\rho \cosh \rho (\pi - s)}{\sinh \rho\pi} \int_0^s e^{-\rho t} f(t)\, dt \right| \leq \frac{\rho \cosh \rho (\pi - s)}{\sinh \rho\pi} \int_0^s |f(t)|\, dt\,; \quad . \quad . \quad (15)$$

hence we see that

$$\frac{\rho \cosh \rho (\pi - s)}{\sinh \rho\pi} \int_0^s e^{-\rho t} f(t)\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

converges to zero, as $\lambda$ tends to $-\infty$. It follows that

$$\frac{\rho \cosh \rho (\pi - s)}{\sinh \rho\pi} \int_0^s \cosh \rho t\, f(t)\, dt \underset{\lambda \to -\infty}{=\!=\!=} \frac{\rho \cosh \rho (\pi - s)}{2 \sinh \rho\pi} \int_0^s e^{\rho t} f(t)\, dt.$$

Since a corresponding result applies to the second member of (14), we finally obtain

$$-\lambda \int_0^\pi \Gamma_\lambda (s, t) f(t)\, dt \underset{\lambda \to -\infty}{=\!=\!=} \frac{\rho \cosh \rho (\pi - s)}{2 \sinh \rho\pi} \int_0^s e^{\rho t} f(t)\, dt + \frac{\rho \cosh \rho s}{2 \sinh \rho\pi} \int_s^\pi e^{\rho (\pi - t)} f(t)\, dt. \quad . \quad (17)$$

§ 10. Let us now suppose that $0 < s < \pi$ If $\alpha$ is an arbitrarily assigned positive number less than, or equal to, $s$, the first term of the right-hand member of (17) may be written

$$\frac{\rho \cosh \rho (\pi - s)}{2 \sinh \rho\pi}\left[\int_0^{s-\alpha} e^{\rho t} f(t)\, dt + \int_{s-\alpha}^{s} e^{\rho t} f(t)\, dt\right].$$

Since

$$\left|\frac{\rho \cosh \rho (\pi - s)}{2 \sinh \rho\pi}\int_0^{s-\alpha} e^{\rho t} f(t)\, dt\right| \leq \frac{\rho e^{\rho(s-\alpha)} \cosh \rho (\pi - s)}{2 \sinh \rho\pi}\int_0^{s-\alpha} |f(t)|\, dt,$$

it is clear that

$$\frac{\rho \cosh \rho (\pi - s)}{2 \sinh \rho\pi}\int_0^{s-\alpha} e^{\rho t} f(t)\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

converges to zero, as $\lambda$ tends to $-\infty$.

Again, by a simple substitution, it is easily seen that

$$\int_{s-\alpha}^{s} e^{\rho t} f(t)\, dt = e^{\rho s}\int_0^{\alpha} e^{-\rho t} f(s-t)\, dt\,;$$

and evidently

$$\frac{\rho e^{\rho s} \cosh \rho (\pi - s)}{2 \sinh \rho\pi}\int_0^{\alpha} e^{-\rho t} f(s-t)\, dt \xrightarrow[\lambda \to -\infty]{} \frac{\rho}{2}\int_0^{\alpha} e^{-\rho t} f(s-t)\, dt. \quad . \quad . \qquad (19)$$

We thus prove that

$$\frac{\rho \cosh \rho (\pi - s)}{2 \sinh \rho\pi}\int_0^{s} e^{-\rho t} f(t)\, dt \xrightarrow[\lambda \to -\infty]{} \frac{\rho}{2}\int_0^{\alpha} e^{-\rho t} f(s-t)\, dt. \quad . \quad . \quad . \qquad (20)$$

The restriction $\alpha \leq s$ may now be removed, provided that the function $f(s)$ is defined for values of $s$ outside $(0, \pi)$ in any manner consistent with the condition, that $f(s)$ should have a Lebesgue integral in every finite interval. For if $\alpha > s$, we have

$$\frac{\rho}{2}\int_0^{\alpha} e^{-\rho t} f(s-t)\, dt \xrightarrow[\lambda \to -\infty]{} \frac{\rho}{2}\int_0^{s} e^{-\rho t} f(s-t)\, dt,$$

and the right-hand member of this has been shown to differ from the left-hand member of (20) by a number which tends to zero, as $\lambda$ tends to $-\infty$.

By the same method and with the same convention it may be shown that

$$\frac{\rho \cosh \rho s}{2 \sinh \rho\pi}\int_s^{\pi} e^{\rho(\pi - t)} f(t)\, dt \xrightarrow[\lambda \to -\infty]{} \frac{\rho}{2}\int_0^{\alpha} e^{-\rho t} f(s+t)\, dt. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

Hence, from (17) and (20) we see that

$$-\lambda\int_0^{\pi} K_\lambda (s, t) f(t)\, dt \xrightarrow[\lambda \to -\infty]{} \frac{\rho}{2}\int_0^{\alpha} e^{-\rho t}\left[f(s-t) + f(s+t)\right] dt,$$

for all positive values of $\alpha$, and for all values of $s$ which belong to the open interval $(0, \pi)$. In the paragraphs which immediately follow we shall obtain a more general form for the right-hand member.

§ 11. Let $\chi_1(t)$ be a function of $t$ defined in an interval $(0, \eta)$ $(\eta > 0)$ which possesses a limited derivative of the second order; further, let the function be such that

$$\chi_1(0) = 0, \qquad \chi'_1(0) = 1,$$
$$\chi'_1(t) > 0, \qquad (0 \leqq t \leqq \eta).$$

We propose to prove that $(\alpha \leqq \eta)$

$$\rho \int_0^\alpha e^{-\rho t} f(s-t)\, dt \overline{\overline{\lambda \to -\infty}}\, \rho \int_0^\alpha e^{-\rho t} f[s-\chi_1(t)]\, dt. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In the first place, we observe that, if $n$ is any positive number less than unity and $\rho > 1$,

$$\left| \rho \int_{\frac{\alpha}{\rho^n}}^\alpha e^{-\rho t} f[s-\chi_1(t)]\, dt \right| \leqq \rho e^{-\rho^{1-n}\alpha} \int_0^\alpha |f[s-\chi_1(t)]|\, dt.$$

Since the right-hand member converges to zero as $\lambda$ tends to $-\infty$, it follows that

$$\rho \int_0^\alpha e^{-\rho t} f[s-\chi_1(t)]\, dt \overline{\overline{\lambda \to -\infty}}\, \rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} f[s-\chi_1(t)]\, dt. \quad . \quad . \quad . \quad . \quad (23)$$

Again, by a known theorem of the differential calculus,

$$\chi'_1(t) = 1 + t\chi''_1(t_1) \quad (0 \leqq t \leqq \eta),$$

where $t_1$ is a point of the interval $(0, t)$. Denoting by $c$ the upper limit of $|\chi''_1(t)|$ in $(0, t)$ we see that

$$|\chi'_1(t) - 1| \leqq ct.$$

Hence we have

$$\left| \rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} f[s-\chi_1(t)]\, \chi'_1(t)\, dt - \rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} f[s-\chi_1(t)]\, dt \right| \leqq c\rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} |f[s-\chi_1(t)]|\, t\, dt.$$

Since

$$\rho t e^{-\rho t} \leqq e^{-1},$$

for all values of $\rho$ and $t$ which are not negative, the right-hand member is not greater than

$$ce^{-1} \int_0^{\frac{\alpha}{\rho^n}} |f[s-\chi_1(t)]|\, dt;$$

and this, by a known property of Lebesgue integrals, converges to zero as $\rho$ tends to $\infty$. Referring to (23) we thus see that

$$\rho \int_0^\alpha e^{-\rho t} f[s-\chi_1(t)]\, dt \overline{\overline{\lambda \to -\infty}}\, \rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} f[s-\chi_1(t)]\, \chi'_1(t)\, dt.$$

For values of $\rho$ which are sufficiently great,

$$\left| \rho \int_{\chi_1\left(\frac{\alpha}{\rho^n}\right)}^\alpha e^{-\rho t} f(s-t)\, dt \right| \leqq \rho e^{-\rho^{1-n}\beta} \int_0^\alpha |f(s-t)|\, dt, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where

$$\beta = \rho^n \chi_1\left(\frac{\alpha}{\rho^n}\right).$$

When $\rho$ increases indefinitely $\beta$ clearly converges to $\alpha$; and therefore the right-hand member of (24) converges to zero. It follows that

$$\rho \int_0^{\alpha} e^{-\rho t} f(s-t)\, dt \underset{\lambda \to -\infty}{=\!=} \rho \int_0^{\chi_1\left(\frac{\alpha}{\rho^n}\right)} e^{-\rho t} f(s-t)\, dt.$$

Substituting $t = \chi_1(w)$ in the right-hand member, and then replacing $w$ by $t$, we see that this may be written

$$\rho \int_0^{\alpha} e^{-\rho t} f(s-t)\, dt \underset{\lambda \to -\infty}{=\!=} \rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho \chi_1(t)} f[s-\chi_1(t)]\, \chi'_1(t)\, dt.$$

Taking this in conjunction with the result previously obtained, we see that (22) will be established when it has been proved that

$$\rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} f[s-\chi_1(t)]\, \chi'_1(t)\, dt \underset{\lambda \to -\infty}{=\!=} \rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho \chi_1(t)} f[s-\chi_1(t)]\, \chi'_1(t)\, dt. \quad . \quad . \quad (25)$$

§ 12. We have

$$\left| \rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} f[s-\chi_1(t)]\, \chi'_1(t)\, dt - \rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho \chi_1(t)} f[s-\chi_1(t)]\, \chi'_1(t)\, dt \right|$$
$$\leq \rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} \left|1 - e^{-\rho[\chi_1(t)-t]}\right| \left|f[s-\chi_1(t)]\right| \chi'_1(t)\, dt. \quad . \quad . \quad (26)$$

Now, by a known theorem of the differential calculus,

$$\chi_1(t) = t + \tfrac{1}{2} t^2 \chi''_1(t_2) \quad (0 \leq t \leq \eta),$$

where, as above, $t_1$ is a point of the interval $(0, t)$.

From this we see that

$$|\chi_1(t) - t| \leq \tfrac{1}{2} c t^2 \quad (0 \leq t \leq \eta);$$

and hence that

$$\left|1 - e^{-\rho[\chi_1(t)-t]}\right| \leq e^{\frac{1}{2} c \rho t^2} - 1 \quad (0 \leq t \leq \eta).$$

Thus the right-hand member of (26) is

$$\leq \int_0^{\frac{\alpha}{\rho^n}} \mathrm{P}(\rho, t) \left|f[s-\chi_1(t)]\right| \chi_1'(t)\, dt,$$

where

$$\mathrm{P}(\rho, t) = \rho e^{-\rho t}\left(e^{\frac{1}{2} c \rho t^2} - 1\right) \quad \left(0 \leq t \leq \frac{\alpha}{\rho^n}\right).$$

§ 13. Let us now suppose $n > \frac{1}{2}$; then, for values of $t$ in $\left(0, \frac{\alpha}{\rho^n}\right)$, it is possible to find a positive number such that, for all values of $\rho$ which are greater than it, $\frac{1}{2}c\rho t^2$ is less than unity. For these values of $\rho$ and $t$ we have

$$e^{\frac{1}{2}c\rho t^2} - 1 \leq \frac{\frac{1}{2}c\rho t^2}{1-\frac{1}{2}c\rho t^2},$$

and so

$$\mathrm{P}(\rho, t) \leq \frac{\frac{1}{2}c}{1-\frac{1}{2}c\rho t^2} \cdot \rho^2 t^2 e^{-\rho t}.$$

Since

$$\rho^2 t^2 e^{-\rho t} < 4e^{-2}$$

for values of $\rho$ and $t$ which are not negative, we see from this that, for values of $t$ in $\left(0, \frac{\alpha}{\rho^n}\right)$, $\mathrm{P}(\rho, t)$ is less than a fixed positive number $\overline{\mathrm{P}}$ independent of $\rho$. We therefore have

$$\left|\rho\int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} f[s-\chi_1(t)]\,\chi'_1(t)\,dt - \rho\int_0^{\frac{\alpha}{\rho^n}} e^{-\rho\chi_1(t)} f[s-\chi_1(t)]\,\chi'_1(t)\,dt\right| \leq \overline{\mathrm{P}}\int_0^{\frac{\alpha}{\rho^n}} |f[s-\chi_1(t)]|\,|\chi'_1(t)\,dt.$$

As the right-hand member converges to zero when $\rho$ increases indefinitely, it follows that the relation (25) is true; and hence that

$$\rho\int_0^{\alpha} e^{-\rho t} f(s-t)\,dt \underset{\lambda\to-\infty}{=\!=} \rho\int_0^{\alpha} e^{-\rho t} f[s-\chi_1(t)]\,dt. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

It may be shown in a similar way that

$$\rho\int_0^{\alpha} e^{-\rho t} f(s+t)\,dt \underset{\lambda\to-\infty}{=\!=} \rho\int_0^{\alpha} e^{-\rho t} f[s+\chi_2(t)]\,dt, \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where $\chi_2(t)$ is any function which has a limited second derivation in $(0, \eta)$, and is such that

$$\chi_2(0) = 0, \qquad \chi'_2(0) = 1, \qquad \chi'_2(t) > 0 \qquad (0 \leq t \leq \eta).$$

It follows from the result obtained in § 10 that

$$-\lambda\int_0^{\pi} \mathrm{K}_\lambda(s, t)\,f(t)\,dt \underset{\lambda\to-\infty}{=\!=} \frac{\rho}{2}\int_0^{\alpha} e^{-\rho t}\{f[s-\chi_1(t)] + f[s+\chi_2(t)]\}\,dt. \quad . \quad (28)$$

Let us denote

$$\tfrac{1}{2}\{f[s-\chi_1(t)] + f[s+\chi_2(t)]\},$$

where $s$ is a fixed point of the open interval $(0, \pi)$, by $\mathrm{X}(t)$; and let us suppose, for the moment, that $\overline{\mathrm{X}(+0)}$ is finite. Then, if $\epsilon$ is an arbitrarily assigned positive number, we can choose $\alpha$ so small that

$$\mathrm{X}(t) \leq \overline{\mathrm{X}(+0)} + \epsilon$$

for all values of $t$ in the interval $(0, \alpha)$. We have therefore

$$\rho\int_0^\alpha e^{-\rho t}\mathrm{X}(t)\,dt \leqq [\overline{\mathrm{X}(+0)}+\epsilon]\,\rho\int_0^\alpha e^{-\rho t}\,dt.$$

Since

$$\lim_{\rho\to\infty}\rho\int_0^\alpha e^{-\rho t}\,dt = 1,$$

we see from this that

$$\overline{\lim_{\lambda\to-\infty}}\,-\lambda\int_0^\pi \mathrm{K}_\lambda(s,t)\,f(t)\,dt \leqq \overline{\mathrm{X}(+0)}+\epsilon\,;$$

or, as $\epsilon$ is arbitrarily small,

$$\overline{\lim_{\lambda\to-\infty}}\,-\lambda\int_0^\pi \mathrm{K}_\lambda(s,t)\,f(t)\,dt \leqq \overline{\mathrm{X}(+0)}. \quad . \quad . \quad . \quad (29)$$

This inequality is obviously true when $\overline{\mathrm{X}(+0)}$ has the improper value $+\infty$, provided that we interpret it as suggested in II., § 4; and the reader will be able to prove that it also holds when $\overline{\mathrm{X}(+0)}$ is $-\infty$. It follows that the inequality as written above holds in all cases.

It may be shown in a similar way that

$$\underline{\lim_{\lambda\to-\infty}}\,-\lambda\int_0^\pi \mathrm{K}_\lambda(s,t)\,f(t)\,dt \leqq \underline{\mathrm{X}(+0)}. \quad . \quad . \quad . \quad (30)$$

The function $\mathrm{X}(t)$ depends upon $\chi_1(t)$, $\chi_2(t)$, and so $\overline{\mathrm{X}(+0)}$ may have different values when we replace these functions by others satisfying the requirements of §§ 11, 12. Let us denote by $\overline{\omega(s)}$ the lower limit of the set of values of $\overline{\mathrm{X}(+0)}$ obtained by taking all possible pairs of functions $\chi_1(t)$, $\chi_2(t)$; and let $\underline{\omega(s)}$ be the upper limit of the corresponding values of $\underline{\mathrm{X}(+0)}$. We shall speak of $\overline{\omega(s)}$ as the *upper bilateral limit* of $f(s)$ at the point $s$, and of $\underline{\omega(s)}$ as the *lower bilateral limit* of $f(s)$ at this point. When $\overline{\omega(s)}$ is equal to $\underline{\omega(s)}$, the common value will be referred to as *the bilateral limit* of $f(s)$ at the point $s$, and will be denoted by $\omega(s)$.

It will be obvious from (29) and (30) that

$$\overline{\omega(s)} \geqq \overline{\lim_{\lambda\to-\infty}}\,-\lambda\int_0^\pi \mathrm{K}_\lambda(s,t)\,f(t)\,dt \geqq \underline{\lim_{\lambda\to-\infty}}\,-\lambda\int_0^\pi \mathrm{K}_\lambda(s,t)\,f(t)\,dt \geqq \underline{\omega(s)}. \quad . \quad (31)$$

In particular, it will be clear that, when the bilateral limit of $f(s)$ at the point $s$ exists,

$$\lim_{\lambda\to-\infty}\,-\lambda\int_0^\pi \mathrm{K}_\lambda(s,t)\,f(t)\,dt \quad . \quad . \quad . \quad . \quad (32)$$

exists, and is equal to it.

From the inequality just written, and the definitions given above, we have

$$\overline{\mathrm{X}(+0)} \geqq \overline{\omega(s)} \geqq \underline{\omega(s)} \geqq \underline{\mathrm{X}(+0)}. \quad . \quad . \quad . \quad . \quad (33)$$

It follows that, if $\chi_1(t)$, $\chi_2(t)$ can be chosen in such a way that

$$X(+0) = \lim_{t \to 0} \tfrac{1}{2} \{ f[s-\chi_1(t)] + f[s+\chi_2(t)] \}$$

exists, then the bilateral limit of $f(s)$ at the point $s$ exists, and is equal to it. In particular, the bilateral limit of $f(s)$ exists at the point $s$, whenever

$$\lim_{t \to 0} \tfrac{1}{2} [f(s-t) + f(s+t)]$$

exists.

§ 14. In the preceding paragraphs we have developed the theory of upper and lower bilateral limits in a form which is adapted to our immediate requirements, but, on reviewing §§ 11–13, the reader will find that, so far as concerns the definition of these numbers and the fundamental inequality (33), $f(s)$ may be any function which has a Lebesgue integral in an interval $(a, b)$, and $s$ any point lying within this interval. The definitions given are clearly applicable to the more general case, as also are the relations (22) and (27). Choosing a fixed positive number A such that $a \leq s-\mathrm{A}$, $b \geq s+\mathrm{A}$, the reader will easily prove that

$$\frac{\rho}{2} \int_0^{\mathrm{A}} e^{-\rho t} [f(s-t) + f(s+t)]\, dt \xrightarrow[\lambda \to -\infty]{} \frac{\rho}{2} \int_0^{a} e^{-\rho t} [f(s-t) + f(s+t)]\, dt.$$

Hence (28) may be adapted for the more general case by substituting

$$\frac{\rho}{2} \int_0^{\mathrm{A}} e^{-\rho t} [f(s-t) + f(s+t)]\, dt$$

in place of the left-hand member. Proceeding as above, we then obtain

$$\overline{\omega(s)} \geq \overline{\lim_{\rho \to \infty}} \frac{\rho}{2} \int_0^{\mathrm{A}} e^{-\rho t} [f(s-t) + f(s+t)]\, dt \geq \underline{\lim_{\rho \to \infty}} \frac{\rho}{2} \int_0^{\mathrm{A}} e^{-\rho t} [f(s-t) + f(s+t)]\, dt \geq \underline{\omega(s)} \quad (34)$$

in place of (31). It now follows that the fundamental inequality (33) is valid at each point $s$ of the open interval $(a, b)$.

We may deduce from (33) an important property of functions which are integrable in $(a, b)$ in accordance with the definition of Lebesgue. For, from it, it will be clear that *when*

$$\lim_{t \to 0} \tfrac{1}{2} \{ f[s-\chi_1(t)] + f[s+\chi_2(t)] \}$$

*exists at a point of the open interval* $(a, b)$, *it has a value independent of* $\chi_1(t)$ *and* $\chi_2(t)$, namely, the bilateral limit of $f(s)$ at $s$.

It is not consonant with the plan of the present memoir to pursue this topic further.

§ 15. We have now to consider the behaviour of

$$-\lambda \int_0^{\pi} \mathrm{K}_\lambda(s, t) f(t)\, dt$$

as λ tends to $-\infty$, when $s$ is one of the end points of $(0, \pi)$; let us suppose, in the first place, that $s = 0$. From the results obtained in §§ 8, 9, we see that

$$-\lambda \int_0^\pi K_\lambda(0, t) f(t)\, dt \underset{\lambda \to -\infty}{\overline{\underline{\quad}}} \frac{\rho}{2 \sinh \rho\pi} \int_0^\pi e^{\rho(\pi - t)} f(t)\, dt.$$

The right-hand member is

$$\frac{\rho e^{\rho\pi}}{2 \sinh \rho\pi} \left[ \int_0^\alpha e^{-\rho t} f(t)\, dt + \int_\alpha^\pi e^{-\rho t} f(t)\, dt \right],$$

where $\alpha$ is any positive number less than, or equal to, $\pi$. Proceeding as in § 10, it may be shown that

$$\lim_{\rho \to \infty} \frac{\rho e^{\rho\pi}}{2 \sinh \rho\pi} \int_\alpha^\pi e^{-\rho t} f(t)\, dt = 0,$$

and that

$$\frac{\rho e^{\rho\pi}}{2 \sinh \rho\pi} \int_0^\alpha e^{-\rho t} f(t)\, dt \underset{\lambda \to -\infty}{\overline{\underline{\quad}}} \rho \int_0^\alpha e^{-\rho t} f(t)\, dt.$$

Hence we obtain the result

$$-\lambda \int_0^\pi K_\lambda(0, t) f(t)\, dt \underset{\lambda \to -\infty}{\overline{\underline{\quad}}} \rho \int_0^\alpha e^{-\rho t} f(t)\, dt.*$$

From this it follows that

$$\overline{f(0+0)} \geq \overline{\lim_{\lambda \to -\infty}} -\lambda \int_0^\pi K_\lambda(0, t) f(t)\, dt \geq \underline{\lim_{\lambda \to -\infty}} -\lambda \int_0^\pi K_\lambda(0, t) f(t)\, dt \geq \underline{f(0+0)};$$

and, in particular, that

$$\lim_{\lambda \to -\infty} -\lambda \int_0^\pi K_\lambda(0, t) f(t)\, dt = f(0+0),$$

whenever the right-hand member exists.

It may be shown in a similar way that

$$\overline{f(\pi-0)} \geq \overline{\lim_{\lambda \to -\infty}} -\lambda \int_0^\pi K_\lambda(\pi, t) f(t)\, dt \geq \underline{\lim_{\lambda \to -\infty}} -\lambda \int_0^\pi K_\lambda(\pi, t) f(t)\, dt \geq \underline{f(\pi-0)},$$

§ 16. It will be observed that so far we have been concerned with the limit (32) for a fixed value of $s$, and that, in consequence, the question of uniform convergence has been left aside. We now proceed to prove that :—

*If the set of points at which $f(s)$ is continuous includes a closed interval $(\gamma, \delta)$ lying wholly within $(0, \pi)$, then, as λ tends to $-\infty$,*

$$-\lambda \int_0^\pi K_\lambda(s, t) f(t)\, dt,$$

*converges uniformly to $f(s)$ in this interval.*

* It is evident that the restriction $\alpha \leq \pi$ may be removed, if we adopt the convention of § 10 in regard to the definition of $f(s)$ outside $(0, \pi)$.

We commence with the equation

$$-\lambda\int_0^\pi [K_\lambda(s,t)-\Gamma_\lambda(s,t)]\,f(t)\,dt = \rho\int_0^\pi \Gamma_\lambda(s,t)\,\alpha(\rho,s,t)\,f(t)\,dt.$$

It was shown in § 8 that $\rho\Gamma_\lambda(s,t)$ is limited, and that, as $\rho$ tends to $\infty$, its limit is zero for unequal values of $s$ and $t$. The function $\rho\Gamma_\lambda(s,t)\,\alpha(\rho,s,t)$ will therefore satisfy the requirements of the theorem of I., § 3, if it can be proved that $\rho\Gamma_\lambda(s,t)$ converges uniformly to zero, for values of $s$ and $t$ such that $|t-s|$ is not less than an arbitrarily assigned positive number $\eta$. That this is so follows at once from the inequality

$$\rho\Gamma_\lambda(s,t) \leq \frac{\cosh\rho(\pi-\eta)}{\sinh\rho\pi} \qquad (|t-s|>\eta),$$

which the reader will establish without difficulty. We deduce that, as $\lambda$ tends to $-\infty$,

$$-\lambda\int_0^\pi [K_\lambda(s,t)-\Gamma_\lambda(s,t)]\,f(t)\,dt$$

converges uniformly to zero in $(0,\pi)$.

Let us now suppose that $\eta$ is any positive number less than the least of $\gamma$, $\pi-\delta$. Referring to the equation (15), it is evident that the left-hand member is less than

$$\frac{\rho\cosh\rho(\pi-\eta)}{\sinh\rho\pi}\int_0^\delta |f(t)|\,dt,$$

for all values of $s$ lying in $(\gamma,\delta)$. It follows that (16) converges uniformly to zero in $(\gamma,\delta)$. In the same manner it may be shown that (18) and the difference between the left- and right-hand members of (19) both converge uniformly to zero in this interval, for a fixed value of $\alpha$.* Hence it appears that the difference between

$$\rho\int_0^s \Gamma_\lambda(s,t)\,f(t)\,dt \qquad \text{and} \qquad \frac{\rho}{2}\int_0^\alpha e^{-\rho t} f(s-t)\,dt$$

converges uniformly to zero in $(\gamma,\delta)$, as $\lambda$ tends to $-\infty$. Finally, since the same may be proved of the difference between

$$\rho\int_s^\pi \Gamma_\lambda(s,t)\,f(t)\,dt \qquad \text{and} \qquad \frac{\rho}{2}\int_0^\alpha e^{-\rho t} f(s+t)\,dt,$$

we deduce that the difference between

$$-\lambda\int_0^\pi K_\lambda(s,t)\,f(t)\,dt \qquad \text{and} \qquad \frac{\rho}{2}\int_0^\alpha e^{-\rho t}\,[f(s-t)+f(s+t)]\,dt$$

converges uniformly to zero in $(\gamma,\delta)$, as $\lambda$ tends to $-\infty$.

* We assume that $\alpha \leq \eta$.

Now we have

$$\frac{\rho}{2}\int_0^\alpha e^{-\rho t}[f(s-t)+f(s+t)]\,dt-f(s)=\frac{\rho}{2}\int_0^\alpha e^{-\rho t}[f(s-t)+f(s+t)-2f(s)]\,dt-f(s)e^{-\rho\alpha} \quad (35);$$

and, in virtue of our hypothesis as to the continuity of $f(s)$,* it is easily shown that $\alpha$ may be chosen small enough to ensure that

$$|f(s\pm t)-f(s)|<\tfrac{1}{2}\epsilon,$$

for all values of $t$ in $(0, \alpha)$, and for all values of $s$ in $(\gamma, \delta)$, the number $\epsilon$ being positive and arbitrarily assigned. With this choice of $\alpha$ the numerical value of the right-hand member of (35) is less than

$$\tfrac{1}{2}\epsilon+\bar{f}e^{-\rho\alpha},$$

where $\bar{f}$ is the greatest value of $|f(s)|$ in the interval. Hence, since this is less than $\epsilon$ for all values of $\rho$ which are greater than a certain positive number, we see that the left-hand member of (35) converges uniformly to zero, as $\lambda$ tends to $-\infty$. It follows from what was said above that, as $\lambda$ tends to $-\infty$,

$$-\lambda\int_0^\pi K_\lambda(s, t)f(t)\,dt$$

converges uniformly to $f(s)$ in $(\gamma, \delta)$.

§ 17. Using the notation of § 3, we have (*vide* II., 4)

$$-\lambda\int_0^\pi K_\lambda(s, t)f(t)\,dt=\sum_{n=1}\frac{-\lambda}{\lambda_n-\lambda}\psi_n(s)\int_0^\pi\psi_n(t)f(t)\,dt.$$

It will be observed that the coefficient of

$$\psi_n(s)\int_0^\pi\psi_n(t)f(t)\,dt$$

on the right-hand side of this equation involves the corresponding singular value, and that, in consequence, its value depends upon the function $q$ and the constants $h'$, $H'$. The following lemma will enable us to replace the coefficient by another which is independent of both the factors mentioned.

*Lemma*:—

*If* $\lambda_1, \lambda_2, \ldots \lambda_n, \ldots$ *are in increasing order of magnitude, the difference between*

$$\sum_{n=1}\frac{-\lambda}{\lambda_n-\lambda}\psi_n(s)\psi_n(t) \quad \textit{and} \quad \sum_{n=1}\frac{-\lambda}{(n-1)^2-\lambda}\psi_n(s)\psi_n(t) \quad . \quad . \quad (36), (37)$$

*converges uniformly to zero, as* $\lambda$ *tends to* $-\infty$, *for all pairs of values of* $s$ *and* $t$ *lying in the square* $0\leq s\leq\pi$, $0\leq t\leq\pi$.

* To prevent misunderstanding, it may be stated that $f(s)$ is continuous in $(\gamma, \delta)$ and in addition $f(\gamma-0)=f(\gamma)$, $f(\delta+0)=f(\delta)$.

For, with the hypothesis stated, there exists a finite number $\nu$ such that

$$|\lambda_n-(n-1)^2|<\nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

for all values of $n$.* The ratio of the numerical values of the corresponding terms of (36) and (37) thus tends to unity as $n$ is increased indefinitely. It follows that (37) is absolutely convergent.

Again

$$\sum_{n=1}\frac{-\lambda}{(n-1)^2-\lambda}\psi_n(s)\psi_n(t)-\sum_{n=1}\frac{-\lambda}{\lambda_n-\lambda}\psi_n(s)\psi_n(t)=\sum_{n=1}\frac{-\lambda[\lambda_n-(n-1)^2]}{[(n-1)^2-\lambda](\lambda_n-\lambda)}\psi_n(s)\psi_n(t) \quad (39).$$

For negative values of $\lambda$ the numerical values of the terms of the series on the right are less than the corresponding terms of

$$\nu\sum_{n=1}\frac{|\psi_n(s)\psi_n(t)|}{\lambda_n-\lambda}.$$

Since we have

$$2|\psi_n(s)\psi_n(t)|\leqq[\psi_n(s)]^2+[\psi_n(t)]^2,$$

it is thus clear that, as $\lambda$ tends to $-\infty$, the left-hand member of (39) will converge uniformly to zero, if

$$\sum_{n=1}\frac{[\psi_n(s)]^2+[\psi_n(t)]^2}{\lambda_n-\lambda}$$

has this property. But

$$\sum_{n=1}\frac{[\psi_n(s)]^2}{\lambda_n-\lambda}=K_\lambda(s,s),$$

and therefore steadily diminishes to zero as $\lambda$ tends to $-\infty$.† It follows from DINI's theorem that $K_\lambda(s,s)$ converges uniformly to zero, for values of $s$ in $(0,\pi)$. Hence the left-hand member of (39) converges uniformly to zero in the square $0\leqq s\leqq\pi$, $0\leqq t\leqq\pi$, and the lemma is established.

§ 18. From this it appears that the difference between (36) and (37) satisfies the requirements of the theorem of I., § 3. Hence the difference between

$$\sum_{n=1}\frac{-\lambda}{\lambda_n-\lambda}\psi_n(s)\int_0^\pi\psi_n(t)f(t)\,dt \quad\text{and}\quad \sum_{n=1}\frac{-\lambda}{(n-1)^2-\lambda}\psi_n(s)\int_0^\pi\psi_n(t)f(t)\,dt \quad . \quad (40)$$

converges to zero, as $\lambda$ tends to $-\infty$, uniformly for values of $s$ in $(0,\pi)$. It follows that the results obtained in §§ 13, 15, 16, remain true when

$$-\lambda\int_0^\pi K_\lambda(s,t)f(t)\,dt$$

is replaced by (40). We have thus established the theorem :—

* *Vide* IV., § 5.

† "Functions of Positive and Negative Type and their Connection with the Theory of Integral Equations," 'Phil. Trans. Roy. Soc.,' A, vol. 209, pp. 443, 444.

*Let $f(s)$ be any function which has a Lebesgue integral in the interval $(0, \pi)$. Let $\psi_1(s), \psi_2(s), \ldots, \psi_n(s), \ldots$, be the complete system of normal functions which, for suitable values of $\mu$, satisfy the differential equation*

$$\frac{d^2u}{ds^2}+(q+\mu)u = 0$$

*and the boundary conditions*

$$\frac{du}{ds}-h'u = 0 \quad \text{at} \quad s = 0, \qquad \frac{du}{ds}+H'u = 0 \quad \text{at} \quad s = \pi;$$

*further, let the arrangement of these functions be such that the corresponding values of $\mu$ increase with $n$. Then, as $\lambda$ tends to $-\infty$,*

$$\sum_{n=1} \frac{-\lambda}{(n-1)^2-\lambda}\psi_n(s)\int_0^\pi \psi_n(t)f(t)\,dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

*converges to the bilateral limit of $f(s)$ at each point of the open interval $(0, \pi)$ where this limit exists as a finite number; moreover, at a point where the bilateral limit has one of the improper values $\pm\infty$, it diverges to this value and is non-oscillatory. If the set of points at which $f(s)$ is continuous includes a closed interval lying within $(0, \pi)$, then (40) converges uniformly to $f(s)$ in this interval. At the end point $\begin{matrix}0\\\pi\end{matrix}$, (40) converges to $\begin{matrix}f(0+0)\\f(\pi-0)\end{matrix}$, when this limit exists as a finite number; further, when either of these limits has one of the improper values $\pm\infty$, (40) diverges to this value, and is non-oscillatory at the corresponding end point.*

As a corollary to this theorem it should be observed that, *when*

$$\lim_{t\to 0}\tfrac{1}{2}[f(s-t)+f(s+t)]$$

*exists as a finite number at a point of the open interval $(0, \pi)$, (40) converges to this number* (vide § 13).

The reader will recall that the system of normal functions $\psi_1(s), \psi_2(s), \ldots, \psi_n(s), \ldots$ is unaltered when (2) is replaced by

$$\frac{d^2u}{ds^2}+(q+\kappa+\mu)u = 0$$

(*vide* § 3). There is therefore no necessity to suppose that the values of $\mu$ referred to in the enunciation of the above theorem are all positive.

§ 19. It will be convenient to state here how far the preceding results remain true when the pair of boundary conditions for $(a, b)$ is one of the three ${}_a\mathrm{B}$, ${}^b\mathrm{B}$, ${}_a^b\mathrm{B}$, and hence that for $(0, \pi)$ is one of the three ${}_0\mathrm{B}'$, ${}^\pi\mathrm{B}'$, ${}_0^\pi\mathrm{B}'$. In each case the asymptotic formula for $K_\lambda(s, t)$ is of the same character as that obtained in § 7. Thus, when the pair of conditions for $(0, \pi)$ is ${}_0B'$, we have

$$K_\lambda(s, t) = {}_0\Gamma_\lambda(s, t)\left(1+\frac{\alpha(\rho, s, t)}{\rho}\right), \quad \quad (41)$$

where ${}_0\Gamma_\lambda(s, t)$ is the symmetric function of $s$ and $t$ which is such that

$${}_0\Gamma_\lambda(s, t) = \frac{\sinh \rho s \,.\, \cosh \rho(\pi-t)}{\rho \cosh \rho\pi} \, (s \leq t).$$

When the pair of conditions is ${}^\pi B'$, the function ${}_0\Gamma_\lambda(s, t)$ in (41) must be replaced by ${}^\pi\Gamma_\lambda(s, t)$, where ${}^\pi\Gamma_\lambda(s, t)$ is the symmetric function such that

$${}^\pi\Gamma_\lambda(s, t) = \frac{\cosh \rho s \sinh \rho(\pi-t)}{\rho \cosh \rho\pi} \, (s \leq t).*$$

Finally, when the pair of boundary conditions is ${}^\pi_0B'$, it will be seen that ${}_0\Gamma_\lambda(s, t)$ must be replaced by ${}^\pi_0\Gamma_\lambda(s, t)$, where the latter function is symmetric and such that

$${}^\pi_0\Gamma_\lambda(s, t) = \frac{\sinh \rho s \sinh \rho(\pi-t)}{\rho \sinh \rho\pi} \, (s \leq t).$$

From these formulæ it may be deduced that the results obtained in §§ 13, 16 are still applicable. At the end point $s = 0$, it will be found that the first of the inequalities of § 15 is applicable when the boundary condition at this point is

$$\frac{du}{ds} - h'u = 0.$$

When the boundary condition is $u = 0$, we have

$$K_\lambda(0, t) = 0 \quad (0 \leq t \leq \pi),$$

and the inequality is no longer true, save under special circumstances. Corresponding remarks apply to the end point $s = \pi$.

The result obtained in § 17 also requires modification. Using the same notation, when the pair of boundary conditions is either ${}_0B'$, or ${}^\pi B'$, the inequality (38) must be replaced by

$$|\lambda_n - (n-\tfrac{1}{2})^2| < \nu.$$

From this it may be shown that, in both cases, the difference between

$$\sum_{n=1} \frac{-\lambda}{\lambda_n - \lambda} \psi_n(s)\, \psi_n(t), \quad \quad (36)$$

and

$$\sum_{n=1} \frac{-\lambda}{(n-\frac{1}{2})^2 - \lambda} \psi_n(s)\, \psi_n(t)$$

* The reader will perceive that this result may at once be deduced from the former one (*cf.* § 6).

converges uniformly to zero, as $\lambda$ tends to $-\infty$, for all pairs of values of $s$ and $t$ lying in the square $0 \leq s \leq \pi$, $0 \leq t \leq \pi$.

When the pair of boundary conditions is ${}_0^{\pi}B'$, (38) must be replaced by

$$|\lambda_n - n^2| < \nu;$$

hence, in this case, the difference between (36) and

$$\sum_{n=1} \frac{-\lambda}{n^2-\lambda} \psi_n(s)\, \psi_n(t)$$

converges uniformly to zero.

The reader will observe that corresponding changes must be made in the enunciation of the theorem of § 18.

§ 20. Let $\psi_1(s)$, $\psi_2(s)$, ..., $\psi_n(s)$, ... be a complete set of normal functions which satisfy the differential equation

$$\frac{d^2u}{ds^2} + (q+\mu)\, u = 0,$$

and any one of the pairs of boundary conditions $B'$, ${}_0B'$, ${}^{\pi}B'$, ${}_0^{\pi}B'$; further, let the order of these functions be such that the corresponding singular values increase with $n$. Then, if $f(s)$ is any function which has a Lebesgue integral in $(0, \pi)$, the terms of the series

$$\psi_1(s) \int_0^{\pi} \psi_1(t) f(t)\, dt + \psi_2(s) \int_0^{\pi} \psi_2(t) f(t)\, dt + \ldots + \psi_n(s) \int_0^{\pi} \psi_n(t) f(t)\, dt + \ldots \quad . \quad (42)$$

will have a definite order, and the coefficients will each have a meaning. We shall refer to (42) as *a canonical Sturm-Liouville series corresponding to* $f(s)$.

Let $s$ be any point of the open interval $(0, \pi)$. Denoting by $U(s)$ and $L(s)$ the upper and lower limits of indeterminacy of the series (42), the general theorem of II., § 4, shows that

$$U(s) \geq \overline{\lim_{\lambda \to -\infty}} -\lambda \int_0^{\pi} K_{\lambda}(s, t) f(t)\, dt \geq \underline{\lim_{\lambda \to -\infty}} -\lambda \int_0^{\pi} K_{\lambda}(s, t) f(t)\, dt \geq L(s).$$

Also, by the results of §§ 13, 19 we have

$$\overline{\omega(s)} \geq \overline{\lim_{\lambda \to -\infty}} -\lambda \int_0^{\pi} K_{\lambda}(s, t) f(t)\, dt \geq \underline{\lim_{\lambda \to -\infty}} -\lambda \int_0^{\pi} K_{\lambda}(s, t) f(t)\, dt \geq \underline{\omega(s)}.$$

By supposing that $\overline{\omega(s)} = \underline{\omega(s)}$, we obtain the theorem :—

I. *If* $U(s)$ *and* $L(s)$ *are the upper and lower limits of indeterminacy at the point* $s$ *of one of the canonical Sturm-Liouville series corresponding to* $f(s)$, *then*

$$U(s) \geq \omega(s) \geq L(s),$$

*at each point where* $\omega(s)$, *the bilateral limit of* $f(s)$, *exists.*

By supposing that $U(s) = L(s)$, we see that :—

II. *The sum of a canonical Sturm-Liouville series corresponding to $f(s)$ at any point where it converges lies between the upper and lower bilateral limits of $f(s)$ at the point.*

Again, by taking the case in which $\mathrm{U}(s) = \mathrm{L}(s)$, $\overline{\omega(s)} = \underline{\omega(s)}$, and the common value is in each case finite, we have

III. *At any point where the bilateral limit of $f(s)$ exists as a finite number, no canonical Sturm-Liouville series corresponding to $f(s)$ can be convergent and have its sum different from this bilateral limit. In particular, no canonical Sturm-Liouville series corresponding to $f(s)$ can converge and have its sum different from*

$$\lim_{t \to 0} \tfrac{1}{2}[f(s+t)+f(s-t)],$$

*at a point where this limit exists.*

This theorem may, of course, be regarded as included in I. or II. Another particular case which is worthy of remark is that in which $\mathrm{U}(s) = \mathrm{L}(s) = \pm\infty$. Since we can only have $\overline{\underline{\omega(s)}} = \pm\infty$ at a point of infinite discontinuity of $f(s)$, we see that

IV. *A canonical Sturm-Liouville series corresponding to $f(s)$ can only diverge to $+\infty$, or to $-\infty$, and be non-oscillatory at a point of infinite discontinuity of $f(s)$.*

Lastly, it is known* that $|\psi_n(s)|$ is less than a fixed positive number for all values of $n$ and $s$. From the result of II., § 7, we deduce that

V. *At any point where the bilateral limit of $f(s)$ exists as a finite number, each canonical Sturm-Liouville series corresponding to $f(s)$ may be made to converge to this limit by the introduction of suitable brackets.*

Different systems of bracketing may have to be employed at the various points of $(0, \pi)$, but, in virtue of results obtained later,† it will be seen that, at any particular point $s$, the same system will suffice for each canonical Sturm-Liouville series.

It will be observed that the above theorems have been stated only for values of $s$ in the open interval $(0, \pi)$. After what has been said in §§ 15, 19, there will be no difficulty in supplying the results which correspond to I.–V. when $s$ is an end point of the interval. It will be found that, if the boundary condition which is satisfied by the normal function of the series is not $u = 0$, all the above results hold good for $s = 0$, provided that we replace $\overline{\omega(s)}$ by $\overline{f(0+0)}$, $\underline{\omega(s)}$ by $\underline{f(0+0)}$, and $\omega(s)$ by $f(0+0$ wherever necessary; for example, corresponding to I., we have the inequality

$$\mathrm{U}(0) \geq f(0+0) \geq \mathrm{L}(0),$$

when $f(s)$ is such that $f(0+0)$ exists. It is unnecessary to consider the case when

* *Cf.* IV., § 6.

† It is shown in the following section that, as $n$ tends to $\infty$, the difference between the sums of the first $n$ terms of any two canonical Sturm-Liouville series converges to zero at each point of the open interval $(0, \pi)$.

the normal functions of the series satisfy the boundary condition $u = 0$ at $s = 0$, for we then have

$$\mathrm{U}(0) = \mathrm{L}(0) = 0,$$

whatever be the nature of $f(s)$. Similar remarks apply when $s$ has the value $\pi$.

§ 21. The theorems of the preceding paragraph apply to FOURIER'S sine and cosine series, since the latter are particular Sturm-Liouville series. It is not difficult to extend them to the FOURIER'S series

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \frac{1}{\pi}\sin s\int_{-\pi}^{\pi} f(t)\sin t\,dt + \frac{1}{\pi}\cos s\int_{-\pi}^{\pi} f(t)\cos t\,dt$$
$$+\ldots+\frac{1}{\pi}\sin ns\int_{-\pi}^{\pi} f(t)\sin nt\,dt + \frac{1}{\pi}\cos ns\int_{-\pi}^{\pi} f(t)\cos nt\,dt + \ldots \quad . \quad . \qquad (43)$$

corresponding to a function $f(s)$ which has a Lebesgue integral in $(-\pi, \pi)$, The series just written is clearly

$$\frac{1}{2\pi}\int_0^{\pi} [f(t)+f(-t)]\,dt + \frac{1}{\pi}\sin s\int_0^{\pi} [f(t)-f(-t)]\sin t\,dt$$
$$+\ldots+\frac{1}{\pi}\sin ns\int_0^{\pi} [f(t)-f(-t)]\sin nt\,dt + \frac{1}{\pi}\cos ns\int_0^{\pi} [f(t)+f(-t)]\cos nt\,dt + \ldots.$$

Let us in the first place suppose that $s$ is a point of the open interval $(0, \pi)$. It is known* that the difference between the sums of the first $n$ terms of FOURIER'S sine and cosine series corresponding to $[f(s)-f(-s)]$ converges to zero, as $n$ is increased indefinitely. Hence the limits of indeterminacy of (43) are identical with those of

$$\frac{1}{2\pi}\int_0^{\pi}[f(t)+f(-t)]\,dt + \frac{1}{2\pi}\int_0^{\pi} [f(t)-f(-t)]\,dt + \ldots + \frac{1}{\pi}\cos ns\int_0^{\pi} [f(t)-f(-t)]\cos nt\,dt$$
$$+\frac{1}{\pi}\cos ns\int_0^{\pi} [f(t)+f(-t)]\cos nt\,dt + \ldots;$$

and, therefore, in virtue of the fact that the $n^{\text{th}}$ term converges to zero as $n$ tends to $\infty$, with those of

$$\frac{1}{\pi}\int_0^{\pi} f(t)\,dt + \frac{2}{\pi}\cos s\int_0^{\pi} f(t)\cos t\,dt + \ldots + \frac{2}{\pi}\cos ns\int_0^{\pi} f(t)\cos nt\,dt + \ldots.$$

Referring to the first inequality of § 20, we deduce from this that U$(s)$ and L$(s)$, the upper and lower limits of indeterminacy of (43), satisfy the inequality

$$\mathrm{U}(s) \geq \overline{\lim_{\lambda\to-\infty}} -\lambda\int_0^{\pi} \Gamma_\lambda(s,t)\,f(t)\,dt \geq \underline{\lim_{\lambda\to-\infty}} -\lambda\int_0^{\pi} \Gamma_\lambda(s,t)\,f(t)\,dt \geq \mathrm{L}(s)\,;$$

* *Vide* IV., §§ 13–15.

which may be written

$$\mathrm{U}(s) \geq \overline{\lim_{\rho\to\infty}} \frac{\rho}{2}\int_0^\alpha e^{-\rho t}[f(s-t)+f(s+t)]\,dt \geq \underline{\lim_{\rho\to\infty}} \frac{\rho}{2}\int_0^\alpha e^{-\rho t}[f(s-t)+f(s+t)]\,dt \geq \mathrm{L}(s), \quad (44)$$

where $\alpha$ is any positive number (§ 10). It is here supposed that $f(s)$ is defined outside $(-\pi, \pi)$ in such a way that it has a Lebesgue integral in any finite interval. In what follows we shall secure this by the rule

$$f(s+2\pi) = f(s),$$

as in the theory of Fourier's series.

Again, when $s$ lies in the open interval $(-\pi, 0)$, we have $s = -|s|$, $|s|$ being a point of the open interval $(0, \pi)$. Proceeding as before, we now find that the limits of indeterminacy of (43) are identical with those of

$$\frac{1}{2\pi}\int_0^\pi [f(t)+f(-t)]\,dt - \frac{1}{2\pi}\int_0^\pi [f(t)-f(-t)]\,dt + \ldots - \frac{1}{\pi}\cos n|s|\int_0^\pi [f(t)-f(-t)]\cos nt\,dt$$
$$+\frac{1}{\pi}\cos n|s|\int_0^\pi [f(t)+f(-t)]\cos nt\,dt - \ldots;$$

and hence with those of

$$\frac{1}{\pi}\int_0^\pi f(-t)\,dt + \frac{2}{\pi}\cos|s|\int_0^\pi f(-t)\cos t\,dt + \ldots + \frac{2}{\pi}\cos n|s|\int_0^\pi f(-t)\cos nt\,dt + \ldots.$$

It follows that, when $s$ lies in the open interval $(-\pi, 0)$,

$$\mathrm{U}(s) \geq \overline{\lim_{\rho\to\infty}} \frac{\rho}{2}\int_0^\alpha e^{-\rho t}\{f[-(|s|-t)]+f[-(|s|+t)]\}\,dt$$
$$\geq \underline{\lim_{\rho\to\infty}} \frac{\rho}{2}\int_0^\alpha e^{-\rho t}\{f[-(|s|-t)]+f[-(|s|+t)]\}\,dt \geq \mathrm{L}(s),$$

whence it appears that (44) is valid for these values of $s$.

Lastly, at either of the points $-\pi$, $0$, $\pi$, it is evident that the limits of indeterminacy of (43) are identical with those of

$$\frac{1}{\pi}\int_0^\pi \tfrac{1}{2}[f(t)+f(-t)]\,dt + \frac{2}{\pi}\cos|s|\int_0^\pi \tfrac{1}{2}[f(t)+f(-t)]\cos t\,dt$$
$$+\ldots+\frac{2}{\pi}\cos n|s|\int_0^\pi \tfrac{1}{2}[f(t)+f(-t)]\cos nt\,dt + \ldots.$$

Since

$$-\lambda\int_0^\pi \Gamma_\lambda(0, t)\,\tfrac{1}{2}[f(t)+f(-t)]\,dt \underset{\lambda\to-\infty}{\rightleftharpoons} \frac{\rho}{2}\int_0^\alpha e^{-\rho t}[f(t)+f(-t)]\,dt$$

(§ 15), we at once deduce that (44) is valid when $s = 0$. Recalling that $f(s)$ is periodic, the reader will be able to prove that it is also valid when $s$ has either of the values $-\pi$, $\pi$.

Now, if $\overline{\omega(s)}$ and $\underline{\omega(s)}$ are the upper and lower bilateral limits of $f(s)$ at *any* point $s$, we have

$$\overline{\omega(s)} \geqq \overline{\lim_{\rho \to \infty}} \frac{\rho}{2} \int_0^a e^{-\rho t} [f(s-t)+f(s+t)]\, dt \geqq \underline{\lim_{\rho \to \infty}} \frac{\rho}{2} \int_0^a e^{-\rho t} [f(s-t)+f(s+t)]\, dt \geqq \underline{\omega(s)}$$

(§ 14). Further, as it has been shown to hold for all values of $s$ in the closed interval $(-\pi, \pi)$, (44) holds for any value of $s$ whatever. We are thus in a position to state theorems, analogous to those numbered I.–V. in the preceding paragraph, on the behaviour of the FOURIER'S series (43). For example, corresponding to III., we have the theorem

*At any point where the bilateral limit of $f(s)$ exists as a finite number, the* FOURIER'S *series corresponding to $f(s)$, if it converges, has this bilateral limit for its sum. In particular, if the series converges at a point where*

$$\lim_{t \to 0} \tfrac{1}{2} [f(s-t)+f(s+t)]$$

*exists, then this limit is its sum.*

We leave the reader to enunciate the other four theorems, merely remarking that each of them is valid for *unrestricted* values of $s$.*

§ 22. From results which have been obtained above we may deduce theorems concerned with the expansion of a function $F(x)$, which has a Lebesgue integral in $(a, b)$, as an infinite series of the type

$$\Psi_1(x) \int_a^b g(y)\, \Psi_1(y)\, F(y)\, dy + \Psi_2(x) \int_a^b g(y)\, \Psi_2(y)\, F(y)\, dy$$
$$+ \ldots + \Psi_n(x) \int_a^b g(y)\, \Psi_n(y)\, F(y)\, dy + \ldots, \quad . \quad . \quad . \quad (45)\dagger$$

where $\Psi_n(x)$ is the solution of

$$\frac{d}{dx}\left(k \frac{dv}{dx}\right) + (gr - l)\, v = 0,$$

which, for $r = r_n$, satisfies one of the pairs of boundary conditions B, ${}_a$B, ${}^b$B, ${}_a^b$B. It is assumed that the functions $\Psi_n(x)$ are made definite‡ by imposing upon them the conditions

$$\int_a^b g\, [\Psi_n(x)]^2\, dx = 1 \quad (n = 1, 2, \ldots), \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

* These theorems are, of course, more general than those obtained by the methods of FEJÉR and LEBESGUE (*vide* HOBSON, 'The Theory of Functions of a Real Variable,' pp. 707–714).

† It was explained in § 1 that $g$ is a function of $x$; we employ $g(y)$ to denote the same function with the argument $y$ instead of $x$.

‡ There is an ambiguity of sign which, however, is of no consequence.

and that their order of arrangement is that in which $r_n$ increases with $n$. With this understanding we shall call (45) *a Sturm-Liouville series corresponding to* F $(x)$.

Applying the transformation of § 2, we see that

$$\xi^{-1/2}(gk)^{1/4}\Psi_n(x),$$

when expressed in terms of $s$, becomes a function, say $\psi_n(s)$, which satisfies the equation

$$\frac{d^2u}{ds^2}+(q+\mu)u=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the pair of boundary conditions $B'$, ${}_0B'$, ${}^{\pi}B'$, ${}_0^{\pi}B'$ which corresponds to the pair satisfied by $\Psi_n(x)$ : the appropriate value of $\mu$ is evidently $r_n\xi^{-2}$. Since (46) leads to

$$\int_0^{\pi}[\psi_n(s)]^2\,ds=1 \quad (n=1, 2, \ldots),$$

it is thus evident that $\psi_1(s)$, $\psi_2(s)$, ..., $\psi_n(s)$, ... is the complete system of normal functions satisfying (2) and these boundary conditions.

The series (45) becomes

$$\frac{\psi_1(s)}{w(s)}\int_0^{\pi}\psi_1(t)\,w(t)f(t)\,dt+\frac{\psi_2(s)}{w(s)}\int_0^{\pi}\psi_2(t)\,w(t)f(t)\,dt+\ldots+\frac{\psi_n(s)}{w(s)}\int_0^{\pi}\psi_n(t)\,w(t)f(t)\,dt+\ldots, \quad (47)$$

where $f(s)$ is F $(x)$ expressed as a function of $s$, and $w(s)$ has the meaning attached to it in § 2.

§ 23. Let $\Upsilon(x)$ and $\Lambda(x)$ be the upper and lower limits of indeterminacy of the series (45) at the point $x$. These numbers are obviously the upper and lower limits of indeterminacy of the series (47) at the corresponding point $s$ of $(0, \pi)$. From the results of II, § 4, we therefore have

$$\Upsilon(x)\geqq\frac{1}{w(s)}\overline{\lim_{\lambda\rightarrow-\infty}}-\lambda\int_0^{\pi}K_{\lambda}(s,t)\,w(t)f(t)\,dt\geqq\frac{1}{w(s)}\underline{\lim_{\lambda\rightarrow-\infty}}-\lambda\int_0^{\pi}K_{\lambda}(s,t)\,w(t)f(t)\,dt\geqq\Lambda(x),$$

where $K_{\lambda}(s, t)$ has the signification of § 4. Let us suppose for the moment that $x$ is a point of the open interval $(a, b)$. After what was said in §§ 10, 11, 19, it will be clear that

$$-\lambda\int_0^{\pi}K_{\lambda}(s,t)\,w(t)f(t)\,dt \xrightarrow[\lambda\rightarrow-\infty]{} \frac{\rho}{2}\int_0^{\frac{\alpha}{\rho^n}}e^{-\rho t}[w(s-t)f(s-t)+w(s+t)f(s+t)]\,dt.$$

Further,

$$\left|\rho\int_0^{\frac{\alpha}{\rho^n}}e^{-\rho t}[w(s-t)-w(s)]f(s-t)\,dt\right|\leqq e^{-1}d\int_0^{\frac{\alpha}{\rho^n}}|f(s-t)|\,dt,$$

where $d$ is the upper limit of $|w'(s)|$ in $(s, s+\alpha)$.

Hence

$$\rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} w(s-t) f(s-t)\, dt \overline{\overline{\lambda \to -\infty}}\, \rho w(s) \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} f(s-t)\, dt.$$

Since it can be proved in the same way that

$$\rho \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} w(s+t) f(s+t)\, dt \overline{\overline{\lambda \to -\infty}}\, \rho w(s) \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} f(s+t)\, dt,$$

we see that

$$\frac{-\lambda}{w(s)} \int_0^{\pi} \mathrm{K}_\lambda(s, t)\, w(t) f(t)\, dt \overline{\overline{\lambda \to -\infty}}\, \frac{\rho}{2} \int_0^{\frac{\alpha}{\rho^n}} e^{-\rho t} [f(s-t)+f(s+t)]\, dt,$$

that is to say

$$\overline{\overline{\lambda \to -\infty}}\, -\lambda \int_0^{\pi} \mathrm{K}_\lambda(s, t) f(t)\, dt. \quad \text{(48).}$$

We have hitherto supposed that $s$ is not one of the end points of $(0, \pi)$. When this is so, the reader will be able to prove in a similar manner from the formulæ of §§ 15, 19, that the result stated still holds. It follows that, for all values of $x$ in $(a, b)$,

$$\Upsilon(x) \geqq \overline{\lim_{\lambda \to -\infty}}\, -\lambda \int_0^{\pi} \mathrm{K}_\lambda(s, t) f(t)\, dt \geqq \underline{\lim_{\lambda \to -\infty}}\, -\lambda \int_0^{\pi} \mathrm{K}_\lambda(s, t) f(t)\, dt \geqq \Lambda(x). \quad (49)$$

§ 24. Let us again suppose that $x$ is a point of the open interval $(a, b)$. We propose to show that the upper and lower bilateral limits of $\mathrm{F}(x)$ at $x$ are the same as the upper and lower bilateral limits of $f(s)$ at the corresponding point $s$. For, since

$$\mathrm{F}(x) = f\left[\xi \int_a^x \left(\frac{g}{k}\right)^{1/2} dx\right],$$

it follows that, if $\chi_1(t)$, $\chi_2(t)$ are the functions defined in §§ 11, 12, we have

$$\tfrac{1}{2}\{f[s-\chi_1(t)]+f[s+\chi_2(t)]\} = \tfrac{1}{2}\{\mathrm{F}[x-\vartheta_1(y)]+\mathrm{F}[x+\vartheta_2(y)]\}, \quad (50)$$

where $y = \xi^{-1}k^{1/2}g^{-1/2}t$, and the functions $\vartheta_1(y)$, $\vartheta_2(y)$ are defined by

$$\chi_1(t) = \xi \int_{x-\vartheta_1(y)}^{x} \left(\frac{g}{k}\right)^{1/2} dx, \qquad \chi_2(t) = \xi \int_x^{x+\vartheta_2(y)} \left(\frac{g}{k}\right)^{1/2} dx. \quad (51)$$

Since the functions $g$ and $k$ are always positive and possess continuous derivatives in $(a, b)$, it is evident that these relations define $\vartheta_1(y)$ and $\vartheta_2(y)$ as functions of $y$ possessing limited second derivatives in a certain interval $(0, \zeta)\,(\zeta > 0)$. Further, we have

$$\vartheta_1(0) = \vartheta_2(0) = 0, \qquad \vartheta'_1(0) = \vartheta'_2(0) = 1,$$
$$\vartheta'_1(y) > 0, \qquad \vartheta'_2(y) > 0 \qquad (0 \leqq y \leqq \zeta).$$

Denoting by $\overline{\Omega(x)}$ and $\underline{\Omega(x)}$ the upper and lower bilateral limits of $F(x)$ at the point $x$, and employing the notation of § 13, it follows from (50) that

$$\overline{\omega(s)} \geq \overline{\Omega(x)}, \qquad \underline{\omega(s)} \leq \underline{\Omega(x)}.$$

Again, if we commence with *any* two functions $\vartheta_1(y)$, $\vartheta_2(y)$ having the properties above mentioned, it is easily seen that the relations (51) together with $t = \xi k^{-1/2} g^{1/2} y$ define $\chi_1(t)$, $\chi_2(t)$ as functions of $t$ satisfying the requirements laid down in §§ 11, 12. We now deduce from (50) the relations

$$\overline{\omega(s)} \leq \overline{\Omega(x)}, \qquad \underline{\omega(s)} \geq \underline{\Omega(x)}.$$

It follows that we must have

$$\overline{\omega(s)} = \overline{\Omega(x)}, \qquad \underline{\omega(s)} = \underline{\Omega(x)}.$$

§ 25. From this result, and the inequalities established in § 13, we infer that

$$\overline{\Omega(x)} \geq \overline{\lim_{\lambda \to -\infty}} -\lambda \int_0^\pi K_\lambda(s, t) f(t)\, dt \geq \underline{\lim_{\lambda \to -\infty}} -\lambda \int_0^\pi K_\lambda(s, t) f(t)\, dt \geq \underline{\Omega(x)},$$

at any point of the open interval $(a, b)$. Taken in conjunction with the inequalities (49), these at once lead to theorems on the convergence of the general Sturm-Liouville series, corresponding exactly to those numbered I.–V. in § 20. We shall therefore content ourselves with the enunciation of that corresponding to III. This reads as follows:—

*At any point where the bilateral limit of* $F(x)$ *exists as a finite number, no Sturm-Liouville series corresponding to* $F(x)$ *can be convergent and have its sum different from this bilateral limit. In particular, no Sturm-Liouville series corresponding to* $F(x)$ *can converge and have its sum different from*

$$\lim_{y \to 0} \tfrac{1}{2}[F(x-y) + F(x+y)]$$

*at a point where this limit exists.*

As regards the end points of $(a, b)$, we clearly have

$$\overline{F(a+0)} = \overline{f(0+0)}, \qquad \underline{F(a+0)} = \underline{f(0+0)},$$
$$\overline{F(b-0)} = \overline{f(\pi-0)}, \qquad \underline{F(b-0)} = \underline{f(\pi-0)}.$$

Hence, referring to the results of §§ 15, 19, we see that

$$\overline{F(a+0)} \geq \overline{\lim_{\lambda \to -\infty}} -\lambda \int_0^\pi K_\lambda(0, t) f(t)\, dt \geq \underline{\lim_{\lambda \to -\infty}} -\lambda \int_0^\pi K_\lambda(0, t) f(t)\, dt \geq \underline{F(a+0)},$$

when the boundary condition at $x = a$ is

$$\frac{dv}{dx} - hv = 0\,;$$

and that

$$\overline{\mathrm{F}(b-0)} \geq \overline{\lim_{\lambda \to -\infty}} -\lambda \int_0^\pi \mathrm{K}_\lambda(\pi, t) f(t)\, dt \geq \underline{\lim_{\lambda \to -\infty}} -\lambda \int_0^\pi \mathrm{K}_\lambda(\pi, t) f(t)\, dt \geq \underline{\mathrm{F}(b-0)},$$

when the boundary condition at $x = b$ is

$$\frac{dv}{dx} + \mathrm{H}v = 0.$$

From these inequalities, together with (49), we obtain theorems of the usual kind relative to the behaviour of the general Sturm-Liouville series (45) at an end point of $(a, b)$. Thus, for example, when $\mathrm{F}(a+0)$ exists as a finite number, and the boundary condition satisfied by the functions $\Psi_n(x)$ at $x = a$ is not $v = 0$, it will be found that, if the Sturm-Liouville series converges at $x = a$, its sum must be $\mathrm{F}(a+0)$. When the boundary condition satisfied by the functions $\Psi_n(x)$ at $x = a$ is $v = 0$, the terms of the series all vanish and no discussion of the convergence of the series at this point is necessary. Similar remarks apply at the end point $x = b$.

§ 26. In conclusion, it should be observed that the theorem enunciated in § 18 may be stated in a form applicable to the general Sturm-Liouville series. Let us suppose that the functions $\Psi_n(x)$ satisfy the pair of boundary conditions B, and, consequently, that the normal functions $\psi_n(s)$ satisfy B′. By employing the transformation of § 2 (*cf.* § 22), the reader will be able to establish that

$$\sum_{n=1} \frac{-\lambda}{(n-1)^2-\lambda} \Psi_n(x) \int_a^b g(y)\, \Psi_n(y)\, \mathrm{F}(y)\, dy \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52)$$

is equal to

$$-\frac{\lambda}{w(s)} \int_0^\pi \mathrm{K}_\lambda(s, t)\, w(t) f(t)\, dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53)$$

Recalling the relation (48), it follows from the inequalities of the preceding paragraph that, as $\lambda$ tends to $-\infty$, (52) converges to $\Omega(x)$ at each point of the open interval $(a, b)$, that at the end point $a$ it converges to $\mathrm{F}(a+0)$, and that at $b$ it converges to $\mathrm{F}(b-0)$; it being assumed in each case that the limit mentioned exists as a finite number. Moreover, when any one of the limits $\Omega(x)$, $\mathrm{F}(a+0)$, $\mathrm{F}(b-0)$ has an improper value $\pm\infty$, it is plain that (52) diverges to this value and is non-oscillatory at the corresponding point of $(a, b)$. Again, if the set of points at which $\mathrm{F}(x)$ is continuous includes an interval $(a_1, b_1)$ lying within $(a, b)$, the set of points at which $w(s) f(s)$ is continuous includes the corresponding interval of $(0, \pi)$. It follows at once from the theorem of § 16, that (53) converges uniformly to $f(s)$ in the latter interval; and therefore that (52) converges uniformly to $\mathrm{F}(x)$ in $(a_1, b_1)$.

We have thus established the theorem :—

*Let* $\mathrm{F}(x)$ *be any function which has a Lebesgue integral in the interval* $(a, b)$. *Let* $\Psi_1(x)$, $\Psi_2(x)$, … $\Psi_n(x)$, … *be the solutions of*

$$\frac{d}{dx}\left(k \frac{dv}{dx}\right) + (gr - l)\, v = 0$$

*which, for suitable values of r satisfy the pair of boundary conditions*

$$\frac{dv}{dx} - hv = 0 \text{ at } x = a, \qquad \frac{dv}{dx} + Hv = 0 \text{ at } x = b.$$

*Moreover, let these solutions be made definite by imposing on them the conditions*

$$\int_a^b g\,[\Psi_n(x)]^2\,dx = 1 \quad (n = 1, 2, \ldots);$$

*and let the arrangement of them be such that the corresponding values of r increase with n. Then, as λ tends to − ∞,*

$$\sum_{n=1} \frac{-\lambda}{(n-1)^2 - \lambda} \Psi_n(x) \int_a^b g(y)\,\Psi_n(y)\,F(y)\,dy \quad \ldots\ldots\ldots \quad (52)$$

*converges to the bilateral limit of* F $(x)$, *at each point of the open interval* $(a, b)$ *where this limit exists as a finite number; further, at a point where the bilateral limit has one of the improper values* $\pm\infty$, *it diverges to this value and is non-oscillatory If the set of points at which* F $(x)$ *is continuous include a closed interval lying within* $(a, b)$, *then* (52) *converges uniformly to F* $(x)$ *in this interval. At the end point* $\begin{matrix} a \\ b \end{matrix}$ (52) *converges to* $\begin{matrix} F(a+0) \\ F(b-0) \end{matrix}$, *when this limit exists as a finite number; further, when either of these limits has one of the improper values* $\pm\infty$, (52) *diverges to this value and is non-oscillatory at the corresponding end point.*

From this we deduce the corollary that, when

$$\lim_{y \to 0} \tfrac{1}{2}[F(x-y) + F(x+y)]$$

exists as a finite number at a point of the open interval $(a, b)$, (52) converges to this number.

After what was said in § 25 there will be no difficulty in perceiving how the results just stated must be modified when the pair of boundary conditions satisfied by the functions $\Psi_n(x)$ is ${}_a$B, ${}^b$B, or ${}_a^b$B.

## IV.—The Convergence of Sturm-Liouville Series.

§ 1. In this section it is proposed to investigate the convergence of Sturm-Liouville series. It will be recalled that with the notation of III., §§ 2, 3, $\psi_n(s)$ is a solution of

$$\frac{d^2u}{ds^2} + (q+\mu)\,u = 0, \quad \ldots\ldots\ldots\ldots \quad (1)$$

which, for $\mu = \lambda_n$, satisfies the pair of boundary conditions B′, *i.e.*,

$$\frac{du}{ds} - h'u = 0 \text{ at } s = 0, \quad \text{and} \quad \frac{du}{ds} + H'u = 0 \text{ at } s = \pi. \quad (2), (3)$$

*Throughout this section we shall assume that the normal functions $\psi_n(s)$ have such an order that the corresponding singular values $\lambda_n$ increase continually with $n$.*

We proceed, in the first place, to obtain asymptotic formulæ for $\psi_n(s)$ and $\lambda_n$, when $n$ is large. Let $u$ be the solution of (1) which satisfies the conditions $u = 1, \dfrac{du}{ds} = h'$ at $s = 0$, when $\mu$ has the positive value $\tau^2$. Since $u$ satisfies the equation

$$[D^2+\tau^2]\,u = -qu,$$

it is evident that

$$u = c_1 \cos \tau s + c_2 \sin \tau s - [D^2+\tau^2]^{-1}\, qu,$$

where $c_1$ and $c_2$ are constants. Proceeding as in III., § 5, we thus see that

$$u = c_1 \cos \tau s + c_2 \sin \tau s - \frac{1}{\tau}\int_0^s q_1 u_1 \sin \tau (s-s_1)\, ds_1,$$

where $q_1$, $u_1$ are what $q$, $u$ become when $s_1$ is substituted for $s$; and it is easily shown that the conditions satisfied by $u$ and $\dfrac{du}{ds}$ at $s = 0$ give $c_1 = 1, \quad c_2 = \dfrac{h'}{\tau}$, as the appropriate values of the constants. It follows that

$$u = \cos \tau s + \frac{h'}{\tau} \sin \tau s - \frac{1}{\tau}\int_0^s q_1 u_1 \sin \tau (s-s_1)\, ds_1. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Denoting by $\bar{u}$ the upper limit of $|u|$ in the interval $(0, \pi)$, we deduce from this the inequality

$$\bar{u} \leqq \left(1+\frac{h'^2}{\tau^2}\right)^{1/2} + \frac{\bar{u}}{\tau}\int_0^\pi |q_1|\, ds_1,$$

which may be written

$$\bar{u} \leqq \left(1+\frac{h'^2}{\tau^2}\right)^{1/2} \left\{1-\frac{1}{\tau}\int_0^\pi |q_1|\, ds_1\right\}^{-1}.$$

It follows that for values of $\tau$ whose lower limit is greater than $\int_0^\pi |q_1|\, ds_1$, $\bar{u}$ is less than a fixed positive number. Using the notation of III., § 6, we deduce from (4) the formula

$$u = \cos \tau s + \frac{\alpha(\tau, s)}{\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The equation (4) may be written

$$u = \cos \tau s \left(1+\frac{1}{\tau}\int_0^s q_1 u_1 \sin \tau s_1\, ds_1\right) + \frac{\sin \tau s}{\tau}\left(h' - \int_0^s q_1 u_1 \cos \tau s_1\, ds_1\right).$$

Supplying the value of $u$ given by (5) on the right-hand side, we obtain

$$u = \cos \tau s \left(1+\frac{1}{\tau}\int_0^s q_1 \sin \tau s_1 \cos \tau s_1\, ds_1 + \frac{\alpha(\tau, s)}{\tau^2}\right)$$
$$+ \sin \tau s \left[\frac{1}{\tau}\left(h' - \int_0^s q_1 \cos^2 \tau s_1\, ds_1\right) + \frac{\alpha(\tau, s)}{\tau^2}\right]. \quad . \quad . \quad . \quad (6)$$

§ 2. The solution $u$ satisfies the boundary condition (2) by its definition. It is easily seen from (4) that it will also satisfy the boundary condition (3) if

$$\tan \pi\tau = \frac{\mathrm{P}}{\tau - \mathrm{P}'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where

$$\mathrm{P} = h' + \mathrm{H}' - \int_0^\pi q_1 u_1 \left(\cos \tau s_1 - \frac{\mathrm{H}'}{\tau} \sin \tau s_1\right) ds_1, \text{ and } \mathrm{P}' = \frac{\mathrm{H}'h'}{\tau} - \int_0^\pi q_1 u_1 \left(\sin \tau s_1 + \frac{\mathrm{H}'}{\tau} \cos \tau s_1\right) ds_1.$$

Using the formula (5), we see that

$$\mathrm{P} = h' + \mathrm{H}' - \int_0^\pi q_1 \cos^2 \tau s_1 \, ds_1 + \frac{\alpha(\tau)}{\tau}, \qquad \mathrm{P}' = -\int_0^\pi q_1 \sin \tau s_1 \,.\, \cos \tau s_1 \, ds_1 + \frac{\alpha(\tau)}{\tau}.$$

Hence the equation (7) may be written

$$\tan \pi\tau = \frac{1}{\tau}\left(h' + \mathrm{H}' - \int_0^\pi q_1 \cos^2 \tau s_1 \, ds_1\right) + \frac{\alpha(\tau)}{\tau^2}.$$

It is easily seen from this that the large positive roots of (7) are of the form

$$\tau_n = n + \frac{1}{n}\left(\frac{h' + \mathrm{H}' - \int_0^\pi q_1 \cos^2 n s_1 \, ds_1}{\pi}\right) + \frac{\alpha(n)}{n^2}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $n$ is a positive integer.

It will be clear that a positive integer $\bar{n}$ may be chosen great enough to ensure that, for $n \geq \bar{n}$, the numbers $\tau_n$ are roots of (7) which increase continually with $n$. Thus, since (7) is the condition that there should be solutions of (1) satisfying B′,

$$\tau_{\bar{n}}^2, \ \tau_{\bar{n}+1}^2, \ \ldots, \ \tau_n^2, \ \ldots$$

are corresponding singular values arranged in increasing order of magnitude. It follows that, for values of $n$ which are not less than a certain positive integer,

$$\tau_n^2 = \lambda_{n+m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $m$ is a positive or negative integer, or zero. The paragraphs which immediately follow will be devoted to the determination of $m$.

§ 3. Referring to §§ 5, 6 of the previous section, it will be seen that, by using (9) and (10) we obtain

$$\zeta_\lambda(s) = 1 + \frac{h'}{\rho} \tanh \rho s - \frac{1}{\rho} \int_0^s q_1 \frac{\cosh \rho s_1 \sinh \rho(s - s_1)}{\cosh \rho s} ds_1 + \frac{\alpha(\rho, s)}{\rho^2}.$$

Since

$$2 \cosh \rho s_1 \sinh \rho(s - s_1) = \sinh \rho s + \sinh \rho(s - 2s_1),$$

and

$$\int_0^s q_1 \frac{\sinh \rho(s - 2s_1)}{\cosh \rho s} ds_1 = \frac{\alpha(\rho, s)}{\rho},$$

we have

$$\zeta_\lambda(s) = 1 + \frac{1}{\rho} \tanh \rho s \left(h' - \tfrac{1}{2} \int_0^s q_1 \, ds_1\right) + \frac{\alpha(\rho, s)}{\rho^2}.$$

Hence

$$\theta_\lambda(s) = \cosh \rho s \left(1 + \frac{\alpha(\rho, s)}{\rho^2}\right) + \frac{1}{\rho} \sinh \rho s \left(h' - \tfrac{1}{2}\int_0^s q_1\, ds_1\right).$$

Supplying this value of $\theta_\lambda(s)$ in the formula

$$\theta'_\lambda(s) = \rho \sinh \rho s + h' \cosh \rho s - \int_0^s q_1 \theta_\lambda(s_1) \cosh \rho(s - s_1)\, ds_1,$$

it is easily shown that

$$\theta'_\lambda(s) = \rho \sinh \rho s + \cosh \rho s \left(h' - \tfrac{1}{2}\int_0^s q_1\, ds_1 + \frac{\alpha(\rho, s)}{\rho}\right).$$

By employing the device of III., § 6, it may be deduced from these results that

$$\phi_\lambda(s) = \cosh \rho(\pi - s)\left(1 + \frac{\alpha(\rho, s)}{\rho^2}\right) + \frac{1}{\rho} \sinh \rho(\pi - s)\left(\mathrm{H}' - \tfrac{1}{2}\int_s^\pi q_1\, ds_1\right),$$

and that

$$\phi'_\lambda(s) = -\rho \sinh \rho(\pi - s) - \cosh \rho(\pi - s)\left(\mathrm{H}' - \tfrac{1}{2}\int_s^\pi q_1\, ds_1 + \frac{\alpha(\rho, s)}{\rho}\right).$$

It follows from these formulæ that

$$\begin{aligned}\delta(\lambda) &= \phi_\lambda(s)\,\theta'_\lambda(s) - \theta_\lambda(s)\,\phi'_\lambda(s)\\ &= \rho \sinh \rho\pi + \cosh \rho\pi\left(h' + \mathrm{H}' - \tfrac{1}{2}\int_0^\pi q_1\, ds_1 + \frac{\alpha(\rho)}{\rho}\right);\end{aligned}$$

hence

$$\delta(\lambda) = \tfrac{1}{2}\rho e^{\rho\pi}\left(1 + \frac{h' + \mathrm{H}' - \tfrac{1}{2}\int_0^\pi q_1\, ds_1}{\rho} + \frac{\alpha(\rho)}{\rho^2}\right).$$

Again we have

$$\begin{aligned}\theta_\lambda(s)\,\phi_\lambda(s) &= \cosh \rho s \cosh \rho(\pi - s)\left(1 + \frac{\alpha(\rho, s)}{\rho^2}\right) + \frac{\cosh \rho(\pi - s)\sinh \rho s}{\rho}\left(h' - \tfrac{1}{2}\int_0^s q_1\, ds_1\right)\\ &\quad + \frac{\cosh \rho s \sinh \rho(\pi - s)}{\rho}\left(\mathrm{H}' - \tfrac{1}{2}\int_s^\pi q_1\, ds_1\right)\\ &= \tfrac{1}{2}\cosh \rho\pi + \tfrac{1}{2}\cosh \rho(\pi - 2s) + \tfrac{1}{2}\frac{\sinh \rho\pi}{\rho}\left(h' + \mathrm{H}' - \tfrac{1}{2}\int_0^\pi q_1\, ds_1\right)\\ &\quad + \tfrac{1}{2}\frac{\sinh \rho(\pi - 2s)}{\rho}\left(\mathrm{H}' - h' + \tfrac{1}{2}\int_0^s q_1\, ds_1 - \tfrac{1}{2}\int_s^\pi q_1\, ds_1\right) + \frac{\alpha(\rho, s)\cosh \rho s \cosh \rho(\pi - s)}{\rho^2}.\end{aligned}$$

Since the integrals of the fourth and fifth terms between the limits 0 and $\pi$ are both of the form

$$\frac{\alpha(\rho)\cosh \rho\pi}{\rho^2},$$

we see from this that

$$2\int_0^\pi \theta_\lambda(s)\,\phi_\lambda(s)\, ds = \cosh \rho\pi\left(\pi + \frac{\alpha(\rho)}{\rho^2}\right) + \frac{\sinh \rho\pi}{\rho} + \frac{\pi \sinh \rho\pi}{\rho}\left(h' + \mathrm{H}' - \tfrac{1}{2}\int_0^\pi q_1\, ds_1\right),$$

which leads to

$$2\int_0^\pi \theta_\lambda(s)\,\phi_\lambda(s)\,ds = \frac{\pi}{2}\,e^{\rho\pi}\left(1+\frac{h'+\mathrm{H}'-\frac{1}{2}\int_0^\pi q_1\,ds_1}{\rho}+\frac{1}{\pi\rho}+\frac{\alpha(\rho)}{\rho^2}\right).$$

§ 4. Let $\mathrm{D}(\lambda)$ be the determinant of $\kappa(s, t)$, the GREEN'S function of

$$\frac{d^2u}{ds^2}+qu = 0$$

for the boundary conditions B′. Then, in accordance with FREDHOLM'S theory, we have

$$\frac{d}{d\lambda}[\log \mathrm{D}(\lambda)] = -\int_0^\pi \mathrm{K}_\lambda(s, s)\,ds.$$

Recalling that $\lambda = -\rho^2$, we see from the equations (III., 6) that

$$\frac{d}{d\rho}[\log \mathrm{D}(\lambda)] = \frac{2\rho\int_0^\pi \theta_\lambda(s)\,\phi_\lambda(s)\,ds}{\delta(\lambda)};$$

hence, applying the formulæ of the preceding paragraph,

$$\frac{d}{d\rho}[\log \mathrm{D}(\lambda)] = \pi+\frac{1}{\rho}+\frac{\alpha(\rho)}{\rho^2}.$$

Now let $\Delta(\lambda)$ be the determinant of the GREEN'S function of

$$\frac{d^2u}{ds^2}+\kappa u = 0 \quad (\kappa < 0), \qquad (10)$$

for the boundary conditions

$$\frac{du}{ds} = 0 \quad \text{at} \quad s = 0, \qquad \frac{du}{ds} = 0 \quad \text{at} \quad s = \pi. \qquad (11)$$

From the result obtained above we see that

$$\frac{d}{d\rho}[\log \Delta(\lambda)] = \pi+\frac{1}{\rho}+\frac{\alpha(\rho)}{\rho^2}.$$

Thus we have

$$\frac{d}{d\rho}\left(\log\frac{\mathrm{D}(\lambda)}{\Delta(\lambda)}\right) = \frac{\alpha(\rho)}{\rho^2}.$$

If $\lambda_0 = -\rho_0^2$ is any negative number, we obtain from this, by integration,* the formula

$$\frac{\mathrm{D}(\lambda)}{\Delta(\lambda)} = \frac{\mathrm{D}(\lambda_0)}{\Delta(\lambda_0)}\,e^{\int_{\rho_0}^{\rho}\frac{\alpha(\rho)}{\rho^2}d\rho}.$$

Since the integral

$$\int_{\rho_0}^{\rho}\frac{\alpha(\rho)}{\rho^2}\,d\rho$$

tends to a finite limit as $\rho$ tends to $\infty$, we thus see that, *as* $\lambda$ *tends to* $-\infty$, *the quotient* $\mathrm{D}(\lambda)/\Delta(\lambda)$ *tends to a finite limit which is not zero.*

* The function $\alpha(\rho)$ is evidently integrable.

§ 5. The singular values of $\kappa(s, t)$ (III., § 3) being $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$, we have

$$D(\lambda) = \prod_{n=1} \left(1-\frac{\lambda}{\lambda_n}\right),^*$$

which may be written

$$D(\lambda) = \prod_{n=1}^{\bar{n}+m-1} \left(1-\frac{\lambda}{\lambda_n}\right) \prod_{n=\bar{n}} \left(1-\frac{\lambda}{\tau_n^{\,2}}\right),$$

where $m$ has the signification of § 2, and $\bar{n}$ is any positive integer greater than $1-m$, which is such that (9) holds for $n \geq \bar{n}$. Again, the singular values of the GREEN'S function of (10), for the boundary conditions (11), are the values of $\mu$ for which there exist solutions of

$$\frac{d^2u}{ds^2} + (\kappa+\mu)\,u = 0$$

satisfying these boundary conditions; they are therefore $-\kappa$, $1^2-\kappa$, $2^2-\kappa$, ..., $n^2-\kappa$, .... Hence

$$\Delta(\lambda) = \prod_{n=1} \left(1-\frac{\lambda}{(n-1)^2-\kappa}\right) = \prod_{n=1}^{\bar{n}} \left(1-\frac{\lambda}{(n-1)^2-\kappa}\right) \prod_{n=\bar{n}} \left(1-\frac{\lambda}{n^2-\kappa}\right).$$

It follows that

$$\frac{D(\lambda)}{\Delta(\lambda)} = \frac{\prod_{n=1}^{\bar{n}+m-1}\left(1-\dfrac{\lambda}{\lambda_n}\right)}{\prod_{n=1}^{\bar{n}}\left(1-\dfrac{\lambda}{(n-1)^2-\kappa}\right)} P(\lambda), \quad \ldots \ldots \ldots \quad (12)$$

where

$$P(\lambda) = \prod_{n=\bar{n}} \left(\frac{1-\dfrac{\lambda}{\tau_n^{\,2}}}{1-\dfrac{\lambda}{n^2-\kappa}}\right) = \prod_{n=\bar{n}} \left(1-\frac{n^2-\kappa-\tau_n^{\,2}}{n^2-\kappa-\lambda}\right)\left(\frac{n^2-\kappa}{\tau_n^{\,2}}\right). \quad \ldots \quad (13)$$

Now from (8) we see that there exists a positive number $\nu$ such that

$$|\tau_n^{\,2}-n^2| < \nu,$$

for all values of $n \geq \bar{n}$. From this it is easily seen that

$$\prod_{n=\bar{n}} \left(1-\frac{n^2-\kappa-\tau_n^{\,2}}{n^2-\kappa-\lambda}\right)$$

and

$$\prod_{n=\bar{n}} \left(\frac{n^2-\kappa}{\tau_n^{\,2}}\right)$$

are both convergent. Further, recalling that $\kappa$ is negative, we have

$$\left|\prod_{n=\bar{n}} \left(1-\frac{n^2-\kappa-\tau_n^{\,2}}{n^2-\kappa-\lambda}\right)-1\right| \leq \prod_{n=\bar{n}} \left(1+\frac{|n^2-\kappa-\tau_n^{\,2}|}{n^2-\kappa-\lambda}\right)-1 \leq \prod_{n=\bar{n}} \left(1+\frac{\nu-\kappa}{n^2-\kappa-\lambda}\right)-1. \quad (14)$$

* "Functions of Positive and Negative Type and their Connection with the Theory of Integral Equations," 'Phil. Trans. Royal Society,' Series A, vol. 209, p. 445.



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Since the product

$$\prod_{n=\bar{n}} \left(1+\frac{\nu-\kappa}{n^2}\right)$$

is convergent, we may choose a positive integer, $n_1$, greater than $\bar{n}$, which is such that

$$\prod_{n=n_1} \left(1+\frac{\nu-\kappa}{n^2}\right) < 1+\tfrac{1}{2}\epsilon,$$

and therefore that

$$\prod_{n=n_1} \left(1+\frac{\nu-\kappa}{n^2-\kappa-\lambda}\right) < 1+\tfrac{1}{2}\epsilon,$$

for all negative values of $\lambda$.

Also it is clear that we may choose a negative number $\Lambda$, whose numerical value is so great that

$$\prod_{n=\bar{n}}^{n_1-1} \left(1+\frac{\nu-\kappa}{n^2-\kappa-\lambda}\right) < \frac{1+\epsilon}{1+\frac{1}{2}\epsilon}$$

for $\lambda \leqq \Lambda$. Hence, for these values of $\lambda$, we have

$$\prod_{n=\bar{n}} \left(1+\frac{\nu-\kappa}{n^2-\kappa-\lambda}\right) < 1+\epsilon.$$

It follows from (14) that

$$\lim_{\lambda \to -\infty} \prod_{n=\bar{n}} \left(1-\frac{n^2-\kappa-\tau_n^{\,2}}{n^2-\kappa-\lambda}\right) = 1\,;$$

and therefore from (13) that, as $\lambda$ tends to $-\infty$, $P(\lambda)$ tends to a finite limit different from zero.

Since $P(\lambda)$ and $D(\lambda)/\Delta(\lambda)$ both tend to finite limits different from zero, as $\lambda$ tends to $-\infty$, it is obvious from (12) that

$$\lim_{\lambda \to -\infty} \frac{\prod_{n=1}^{\bar{n}+m-1}\left(1-\frac{\lambda}{\lambda_n}\right)}{\prod_{n=1}^{\bar{n}}\left(1-\frac{\lambda}{(n-1)^2-\kappa}\right)}$$

must be finite and different from zero. As this can only be the case when the number of factors in the numerator is equal to the number in the denominator, it is evident that $m = 1$.* Hence from (9) we have

$$\lambda_{n+1} = \tau_n^{\,2}. \qquad (15)$$

As a corollary we deduce from (8) the inequality

$$|\lambda_n-(n-1)^2| < \nu$$

which was employed in III., § 17. This is primarily true for $n$ greater than $\bar{n}$, but, by a suitable choice of $\nu$, it is evidently valid for all values of $n$.

* Previous investigators seem to have overlooked the fact that the value of $m$ is not obvious. They have all tacitly assumed $m = 0$.

§ 6. It follows from (8) and (15) that the large singular values of $\kappa(s, t)$ may be calculated from the asymptotic formula

$$\sqrt{\lambda_{n+1}} = \tau_n = n + \frac{1}{n}\left(\frac{h' + \mathrm{H}' - \int_0^\pi q_1 \cos^2 ns_1\, ds_1}{\pi}\right) + \frac{\alpha(n)}{n^2}.$$

We proceed to obtain an asymptotic formula for $\psi_{n+1}(s)$, the normal function corresponding to $\lambda_{n+1}$.

Let $u_n$ denote the function which $u$ (§ 1) becomes when $\tau_n$ is substituted for $\tau$; then, from the definition of $\tau_n$, it is clear that $u_n$ satisfies the pair of boundary conditions B′. The normal function $\psi_{n+1}(s)$ must therefore be a constant multiple of $u_n$; hence, since $\int_0^\pi [\psi_{n+1}(s)]^2\, ds = 1$, we must have

$$\psi_{n+1}(s) = \frac{u_n}{\sqrt{\int_0^\pi u_n^2\, ds}}. \qquad (16)$$

Now, from (6),

$$u_n = \cos\tau_n s\left(1 + \frac{1}{\tau_n}\int_0^s q_1 \sin\tau_n s_1 \cos\tau_n s_1\, ds_1 + \frac{\alpha(\tau_n, s)}{\tau_n^2}\right) + \sin\tau_n s\left[\frac{1}{\tau_n}\left(h' - \int_0^s q_1 \cos^2\tau_n s_1\, ds_1\right) + \frac{\alpha(\tau_n, s)}{\tau_n^2}\right],$$

while from (8) we see that

$$\cos\tau_n s = \cos ns\left(1 + \frac{\alpha(n)}{n^2}\right) - \sin ns\left[\frac{s}{n}\left(\frac{h' + \mathrm{H}' - \int_0^\pi q_1 \cos^2 ns_1\, ds_1}{\pi}\right) + \frac{\alpha(n)}{n^2}\right],$$

with an analogous formula for $\sin\tau_n s$. It follows that

$$u_n = \cos ns\left(1 + \frac{1}{n}\int_0^s q_1 \sin ns_1 \cos ns_1\, ds_1 + \frac{\alpha(n, s)}{n^2}\right)$$
$$+ \sin ns\left[\frac{1}{n}\left(h' - \frac{s}{\pi}(h' + \mathrm{H}') - \int_0^s q_1 \cos^2 ns_1\, ds_1 + \frac{s}{\pi}\int_0^\pi q_1 \cos^2 ns_1\, ds_1\right) + \frac{\alpha(n, s)}{n^2}\right].$$

From the formula just written we see that

$$(u_n - \cos ns)^2 = \frac{\alpha(n, s)}{n^2}.$$

Integrating between the limits 0 and $\pi$, it will be found that this leads to

$$\int_0^\pi u_n^2\, ds = 2\int_0^\pi u_n \cos ns\, ds - \frac{\pi}{2} + \frac{\alpha(n)}{n^2}. \qquad (17)$$

The function $2u_n \cos ns$ is of the form

$$2\cos^2 ns + \frac{1}{n}\int_0^s q_1 \sin ns_1 \cos ns_1\, ds_1 + \frac{1}{n}[\beta_1(n, s)\cos 2ns + \beta_2(n, s)\sin 2ns] + \frac{\alpha(n, s)}{n^2},$$

where $\beta_1(n, s)$ and $\beta_2(n, s)$ are functions of $n$ and $s$ whose derivatives with respect to

$s$ are both of the form $\alpha(n, s)$ in $(0, \pi)$. By an application of the rule of integration by parts, it is easily seen that

$$\int_0^\pi \beta_1(n, s) \cos 2ns\, ds = \frac{\alpha(n)}{n}, \qquad \int_0^\pi \beta_2(n, s) \sin 2ns\, ds = \frac{\alpha(n)}{n}.$$

Thus we have

$$2\int_0^\pi u_n \cos ns = \pi + \frac{1}{n}\int_0^\pi ds_2 \int_0^{s_2} q_1 \sin ns_1 \cos ns_1\, ds_1 + \frac{\alpha(n)}{n^2},$$

which, taken in conjunction with (17), leads to

$$\int_0^\pi u_n^2\, ds = \frac{\pi}{2} + \frac{1}{n}\int_0^\pi ds_2 \int_0^{s_2} q_1 \sin ns_1 \cos ns_1\, ds_1 + \frac{\alpha(n)}{n^2}.$$

Substituting the positive square root of this value of $\int_0^\pi u_n^2\, ds$, and the value of $u_n$ obtained above, in the right-hand member of (16), we eventually obtain the formula*

$$\psi_{n+1}(s) = \sqrt{\frac{2}{\pi}}\left[\cos ns\left(1 + \frac{A_1(n, s)}{n} + \frac{\alpha(n, s)}{n^2}\right) + \sin ns\left(\frac{A(s) + A_2(n, s)}{n} + \frac{\alpha(n, s)}{n^2}\right)\right], \quad (18)$$

where

$$A(s) = \frac{1}{\pi}\left[(\pi - s)\left(h' - \tfrac{1}{2}\int_0^s q_1\, ds_1\right) - s\left(H' - \tfrac{1}{2}\int_s^\pi q_1\, ds_1\right)\right],$$

$$A_1(n, s) = \frac{1}{2\pi}\int_0^\pi ds_2 \int_{s_2}^s q_1 \sin 2ns_1\, ds_1,$$

and

$$A_2(n, s) = \frac{1}{2\pi}\left[s\int_s^\pi q_1 \cos 2ns_1\, ds_1 - (\pi - s)\int_0^s q_1 \cos 2ns_1\, ds_1\right].$$

Putting in evidence the argument of $q$ as in III., § 6, the reader will see that $\psi_{n+1}(s)$ is the normal function which, for $\mu = \lambda_{n+1}$, satisfies the equation

$$\frac{d^2u}{ds^2} + [q(\pi - s) + \mu]\, u = 0,$$

and the boundary conditions

$$\frac{du}{ds} - H'u = 0 \quad \text{at} \quad s = 0, \qquad \frac{du}{ds} + h'u = 0 \quad \text{at} \quad s = \pi.$$

It follows that $A_1(n, s)$ should remain unaltered when we interchange $h'$ and $H'$, and at the same time substitute $\pi - s$ for $s$, $q(\pi - s_1)$ for $q(s_1)$; also that $A(s)$, $A_2(n, s)$ should merely change sign. We have expressed $A(s)$, $A_1(n, s)$, $A_2(n, s)$ in forms designed to show that they possess these properties.

* It should be observed that there is an ambiguity of sign in the determination of each normal function (*vide* III., § 3). By substituting the positive value of $\sqrt{\int_0^\pi u_n^2\, ds}$ in (16) we obtain the asymptotic formula for that determination of $\psi_{n+1}(s)$ which is positive for $s = 0$. Substituting the negative value of the square root in (16) we obtain the formula for the other determination. This would have served our purpose just as well as (18).

§ 7. The formula (18) is true for all values of $n$ which are greater than, or equal to, a certain positive integer, say N. Let $G(s, t, n)$ denote the sum

$$\sum_{m=N}^{n}\left(\psi_{m+1}(s)\,\psi_{m+1}(t)-\frac{2}{\pi}\cos ms\cos mt\right);$$

then we see from (18) that

$$G(s,t,n)=\frac{2}{\pi}A(s)\sum_{m=N}^{n}\frac{\sin ms\cos mt}{m}+\frac{2}{\pi}A(t)\sum_{m=N}^{n}\frac{\cos ms\sin mt}{m}$$
$$+\frac{2}{\pi}\sum_{m=N}^{n}\left\{\frac{\cos ms\cos mt\,[A_1(m,s)+A_1(m,t)]}{m}+\frac{\sin ms\cos mt A_2(m,s)}{m}\right.$$
$$\left.+\frac{\cos ms\sin mt A_2(m,t)}{m}+\frac{\alpha(m,s,t)}{m_2}\right\}.$$

Now the sum

$$\sum_{m=1}^{n}\frac{\sin mz}{m}$$

is known* to be limited for all values of $z$ and all positive integral values of $n$; further, as $n$ tends to $\infty$, this sum converges uniformly for values of $z$ lying in the closed intervals complementary to the set $(2m\pi-\eta,\ 2m\pi+\eta)$ $(m = 0, \pm 1, \pm 2, \ldots)$, where $\eta$ is any assigned positive number. It follows at once that the sum

$$\sum_{m=N}^{n}\frac{\sin ms\cos mt}{m},$$

being equal to

$$\frac{1}{2}\sum_{m=N}^{n}\frac{\sin m(s+t)}{m}+\frac{1}{2}\sum_{m=N}^{n}\frac{\sin m(s-t)}{m}$$

is limited for $0 \leq s \leq \pi$, $0 \leq t \leq \pi$, $n \geq N$; and that, as $n$ tends to $\infty$, it converges uniformly in those parts of the square $0 \leq s \leq \pi$, $0 \leq t \leq \pi$, which correspond to $|t-s| \geq \eta$. It is easily seen that

$$\sum_{m=N}^{n}\frac{\cos ms\sin mt}{m}$$

has the same property.

Again,

$$\sum_{m=N}^{n}\frac{\cos ms\cos mt\,A_1(m,s)}{m} \quad \ldots \ldots \ldots \quad (19)$$

may be written

$$\frac{1}{2\pi}\int_0^{\pi}ds_2\int_{s_2}^{s}g(s,t,s_1,n)\,q_1\,ds_1, \quad \ldots \ldots \ldots \quad (20)$$

where

$$g(s,t,s_1,n)=\sum_{m=N}^{n}\frac{\cos ms\cos mt\sin 2ms_1}{m}.$$

* For a full discussion, see HOBSON, 'The Theory of Functions of a Real Variable,' pp. 648–643.



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Evidently

$$g(s, t, s_1, n) = \tfrac{1}{4}\left\{\sum_{m=N}^{n}\frac{\sin m(2s_1+s+t)}{m} + \sum_{m=N}^{n}\frac{\sin m(2s_1-s+t)}{m} + \sum_{m=N}^{n}\frac{\sin m(2s_1+s-t)}{m} + \sum_{m=N}^{n}\frac{\sin m(2s_1-s-t)}{m}\right\}.$$

From the remarks made above, it will be seen that $g(s, t, s_1, n)$ is limited for $0 \leqq s \leqq \pi$, $0 \leqq t \leqq \pi$, $0 \leqq s_1 \leqq \pi$, and $n \geqq N$; moreover, if

$$g(s, t, s_1) = \lim_{n \to \infty} g(s, t, s_1, n),$$

$g(s, t, s_1, n)$ converges uniformly to $g(s, t, s_1)$ for those values of $s, t, s_1$, which satisfy the conditions

$$|s_1 \pm \tfrac{1}{2}s \pm \tfrac{1}{2}t| \geqq \eta, \quad |s_1 \pm \tfrac{1}{2}s \pm \tfrac{1}{2}t - \pi| \geqq \eta,$$

and

$$s_1 - \tfrac{1}{2}s - \tfrac{1}{2}t + \pi \geqq \eta.$$

Applying a result obtained in I., § 4, we see that, as $n$ tends to $\infty$, (20) converges uniformly for $0 \leqq s \leqq \pi$, $0 \leqq t \leqq \pi$; and hence that (19) converges uniformly for these values of $s$ and $t$. It may be proved in a similar way that

$$\sum_{m=N}^{n}\frac{\cos ms \cos mt\, A_1(m, t)}{m}$$

converges uniformly for $0 \leqq s \leqq \pi$, $0 \leqq t \leqq \pi$.

§ 8. We have

$$\sum_{m=N}^{n}\frac{\sin ms \cos mt\, A_2(m, s)}{m} = \frac{1}{2\pi}\left\{s\int_s^{\pi} h(s, t, s_1, n)\, q_1\, ds_1 - (\pi - s)\int_0^s h(s, t, s_1, n)\, q_1\, ds_1\right\} \quad (21),$$

where

$$h(s, t, s_1, n) = \tfrac{1}{4}\left\{\sum_{m=N}^{n}\frac{\sin m(2s_1+s+t)}{m} + \sum_{m=N}^{n}\frac{\sin m(2s_1+s-t)}{m} - \sum_{m=N}^{n}\frac{\sin m(2s_1-s+t)}{m} - \sum_{m=N}^{n}\frac{\sin m(2s_1-s-t)}{m}\right\}.$$

It is easily seen that $h(s, t, s_1, n)$, like $g(s, t, s_1, n)$, satisfies the requirements of the theorem of I., § 4; and hence, from the remarks made at the end of this paragraph, that the left-hand member of (21) converges uniformly for $0 \leqq s \leqq \pi$, $0 \leqq t \leqq \pi$. It may be shown in a similar way that

$$\sum_{m=N}^{n}\frac{\cos ms \sin mt\, A_2(m, t)}{m}$$

has the same property.

Lastly, it is at once evident that, as $n$ tends to $\infty$,

$$\sum_{m=N}^{n} \frac{\alpha(m, s, t)}{m^2}$$

converges uniformly for $0 \leq s \leq \pi$, $0 \leq t \leq \pi$.

As a result of the investigations of this paragraph, we now see that $G(s, t, n)$ is limited for $0 \leq s \leq \pi$, $0 \leq t \leq \pi$, $n \geq N$; and that, as $n$ tends to $\infty$, it converges uniformly in those parts of the square $0 \leq s \leq \pi$, $0 \leq t \leq \pi$, which correspond to $|t-s| \geq \eta$.

Consider the function

$$H(s, t, n) = \psi_1(s)\,\psi_1(t) - \frac{1}{\pi} + \sum_{m=1}^{n}\left(\psi_{m+1}(s)\,\psi_{m+1}(t) - \frac{2}{\pi}\cos ms \cos mt\right).$$

For values of $n \geq N$, it differs from $G(s, t, n)$ by

$$\psi_1(s)\,\psi_1(t) - \frac{1}{\pi} + \sum_{m=1}^{N-1}\left(\psi_{m+1}(s)\,\psi_{m+1}(t) - \frac{2}{\pi}\cos ms \cos mt\right).$$

We conclude that $H(s, t, n)$ is limited for $0 \leq s \leq \pi$, $0 \leq t \leq \pi$, $n \geq N$; and that, as $n$ tends to $\infty$, it converges uniformly in those parts of the square $0 \leq s \leq \pi$, $0 \leq t \leq \pi$ which correspond to $|t-s| \geq \eta$. We proceed now to show that, when $t \neq s$,

$$\lim_{n \to \infty} H(s, t, n) = 0.$$

§ 9. The normal functions which satisfy the differential equation

$$\frac{d^2u}{ds^2} + (\lambda+\mu)\,u = 0,$$

and the boundary conditions

$$\frac{du}{ds} = 0 \text{ at } s = 0, \quad \frac{du}{ds} = 0 \text{ at } s = \pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

are clearly

$$\sqrt{\frac{1}{\pi}}, \quad \sqrt{\frac{2}{\pi}}\cos s, \quad \sqrt{\frac{2}{\pi}}\cos 2s, \quad \ldots, \quad \sqrt{\frac{2}{\pi}}\cos ns, \quad \ldots,$$

the corresponding values of $\mu$ being

$$-\lambda, \quad 1^2-\lambda, \quad 2^2-\lambda, \quad \ldots, \quad n^2-\lambda, \quad \ldots$$

Thus (III., § 3) the Green's function of the equation

$$\frac{d^2u}{ds^2} + \lambda u = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

which corresponds to the boundary conditions (11) is

$$-\frac{1}{\lambda\pi} + \sum_{n=1} \frac{1}{n^2-\lambda} \cdot \frac{2}{\pi}\cos ns \cos nt.$$

Again the solution of (22) which satisfies the conditions $u = 1$, $\frac{du}{ds} = 0$, at $s = 0$ is $\cos\sqrt{\lambda}s$; and the solution which satisfies the conditions $u = 1$, $\frac{du}{ds} = 0$, at $s = \pi$ is $\cos\sqrt{\lambda}(\pi - s)$. It follows (*cf.* III., § 4) that the GREEN'S function of (22) which corresponds to the boundary conditions (11) is

$$-\frac{\cos\sqrt{\lambda}s \,.\, \cos\sqrt{\lambda}(\pi-t)}{\sqrt{\lambda}\sin\sqrt{\lambda}\pi}\ (s \leq t), \qquad -\frac{\cos\sqrt{\lambda}t \,.\, \cos\sqrt{\lambda}(\pi-s)}{\sqrt{\lambda}\sin\sqrt{\lambda}\pi}\ (s \geq t).$$

Comparing this with the formula obtained in III., § 7, we see that, as $\lambda = -\rho^2$, the GREEN'S function is $\Gamma_\lambda(s, t)$.

Thus we have

$$\Gamma_\lambda(s, t) = -\frac{1}{\lambda\pi} + \sum_{n=1} \frac{1}{n^2-\lambda}\,\frac{2}{\pi}\cos ns \cos nt.$$

From the lemma of III., § 17, we see that

$$-\lambda K_\lambda(s, t) \underset{\lambda \to -\infty}{=\!=} \sum_{n=1} \frac{-\lambda}{(n-1)^2-\lambda}\,\psi_n(s)\,\psi_n(t),$$

or

$$\underset{\lambda \to -\infty}{=\!=} \psi_1(s)\,\psi_1(t) + \sum_{n=1} \frac{-\lambda}{n^2-\lambda}\,\psi_{n+1}(s)\,\psi_{n+1}(t).$$

Hence, using the result obtained above,

$$-\lambda[K_\lambda(s, t) - \Gamma_\lambda(s, t)] \underset{\lambda \to -\infty}{=\!=} \left(\psi_1(s)\,\psi_1(t) - \frac{1}{\pi}\right) + \sum_{n=1} \frac{-\lambda}{n^2-\lambda}\left(\psi_{n+1}(s)\,\psi_{n+1}(t) - \frac{2}{\pi}\cos ns \cos nt\right). \quad . \quad . \quad (23)$$

Now, when $s \neq t$, the series

$$\psi_1(s)\,\psi_1(t) - \frac{1}{\pi} + \sum_{n=1}\left(\psi_{n+1}(s)\,\psi_{n+1}(t) - \frac{2}{\pi}\cos ns \cos nt\right)$$

has been shown to be convergent (§ 8). It follows by an argument similar to that employed in II., §§ 3, 4, that, as $\lambda$ tends to $-\infty$, the right-hand member of (23) converges to the sum of this series, that is to say to

$$\lim_{n \to \infty} H(s, t, n).$$

Further, the left-hand member is

$$\rho\Gamma_\lambda(s, t)\,\alpha(\rho, s, t),$$

which, for unequal values of $s$ and $t$, has been shown to converge to the limit zero, as $\rho$ tends to $\infty$ (III., § 8). It follows that

$$\lim H(s, t, n) = 0 \quad (s \neq t).$$

§ 10. Let $f(s)$ be any function which possesses a Lebesgue integral in $(0, \pi)$. Recalling the properties of $H(s, t, n)$ which have been proved in the preceding paragraphs, we see from the theorem of I., § 3, that, as $n$ tends to $\infty$,

$$\int_0^\pi H(s, t, n) f(t)\, dt$$

converges uniformly to zero for all values of $s$ in $(0, \pi)$. Denoting by $\sigma_n(s)$ the sum of the first $n$ terms of the series

$$\psi_1(s) \int_0^\pi \psi_1(t) f(t)\, dt + \psi_2(s) \int_0^\pi \psi_2(t) f(t)\, dt + \ldots + \psi_n(s) \int_0^\pi \psi_n(t) f(t)\, dt + \ldots, \quad . \quad (24)$$

and by $\mathfrak{s}_n(s)$ the sum of the first $n$ terms of

$$\frac{1}{\pi}\int_0^\pi f(t)\, dt + \frac{2}{\pi}\cos s \int_0^\pi f(t) \cos t\, dt + \ldots + \frac{2}{\pi}\cos(n-1)s \int_0^\pi f(t)\cos(n-1)t\, dt + \ldots, \quad (25)^*$$

it follows that, as $n$ tends to $\infty$,

$$\sigma_n(s) - \mathfrak{s}_n(s)$$

converges uniformly to zero, for all values of $s$ in $(0, \pi)$.

If $D(s)$ is any limiting point of the set

$$\sigma_1(s),\ \sigma_2(s),\ \ldots,\ \sigma_n(s),\ \ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

we can select an increasing sequence of integers $n_1, n_2, n_3, \ldots, n_m, \ldots$, in such a way that the sequence

$$\sigma_{n_1}(s),\ \sigma_{n_2}(s),\ \ldots,\ \sigma_{n_m}(s),\ \ldots,$$

tends to the limit $D(s)$. It follows from the result obtained above that the sequence

$$\mathfrak{s}_{n_1}(s),\ \mathfrak{s}_{n_2}(s),\ \ldots,\ \mathfrak{s}_{n_m}(s),\ \ldots,$$

also tends to the limit $D(s)$. We have thus proved that each limiting point of the set (26) is also a limiting point of the set

$$\mathfrak{s}_1(s),\ \mathfrak{s}_2(s),\ \ldots,\ \mathfrak{s}_n(s),\ \ldots;$$

in particular, the upper and lower limits of indeterminacy of the two series are identical. It should be observed that this includes the result that, if either of the series (24), (25) is convergent, so also is the other.

Since

$$\sigma_n(s) - \mathfrak{s}_n(s)$$

converges uniformly to zero in the whole of $(0, \pi)$, we see that, if either of the series

* This is, of course, FOURIER's cosine series corresponding to $f(s)$.

(24), (25) converges uniformly in a certain set of points belonging to the interval $(0, \pi)$, so also does the other.

§ 11. Let us next suppose that the pair of boundary conditions for the interval $(0, \pi)$ is ${}_0\mathrm{B}'$. In this case it may be shown that

$$\sum_{m=1}^{n}\left[\psi_m(s)\,\psi_m(t)-\frac{2}{\pi}\sin\left(m-\tfrac{1}{2}\right)s\,\sin\left(m-\tfrac{1}{2}\right)t\right] \quad . \quad . \quad . \quad . \quad . \quad (27)$$

is limited for $0 \leqq s \leqq \pi$, $0 \leqq t \leqq \pi$, and all positive integral values of $n$; and that, as $n$ tends to $\infty$, it converges uniformly in those parts of the square $0 \leqq s \leqq \pi$, $0 \leqq t \leqq \pi$, which correspond to $|t-s| \geqq \eta$. It will be found that the normal functions which satisfy the differential equation

$$\frac{d^2u}{ds^2}+(\lambda+\mu)\,u = 0,$$

and the boundary conditions

$$u = 0 \quad \text{at} \quad s = 0, \qquad \frac{du}{ds} = 0 \quad \text{at} \quad s = \pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

are

$$\sqrt{\frac{2}{\pi}}\sin\tfrac{1}{2}s, \quad \sqrt{\frac{2}{\pi}}\sin\tfrac{3}{2}s, \quad \ldots, \quad \sqrt{\frac{2}{\pi}}\sin\left(n-\tfrac{1}{2}\right)s, \quad \ldots,$$

the corresponding values of $\mu$ being

$$\left(\tfrac{1}{2}\right)^2-\lambda, \ \left(\tfrac{3}{2}\right)^2-\lambda, \ \ldots, \ \left(n-\tfrac{1}{2}\right)^2-\lambda, \ \ldots .$$

The Green's function of

$$\frac{d^2u}{ds^2}+\lambda u = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

for the boundary conditions (28), will be found to be ${}_0\Gamma_\lambda(s, t)$. Hence

$${}_0\Gamma_\lambda(s, t) = \sum_{n=1}\frac{1}{\left(n-\frac{1}{2}\right)^2-\lambda}\,\frac{2}{\pi}\sin\left(n-\tfrac{1}{2}\right)s\,\sin\left(n-\tfrac{1}{2}\right)t.$$

It follows from this and the result quoted in III., § 19, that

$$-\lambda\left[\mathrm{K}_\lambda(s, t)-{}_0\Gamma_\lambda(s, t)\right] \underset{\lambda \to -\infty}{=\!=\!=} \sum_{n=1}\frac{-\lambda}{\left(n-\frac{1}{2}\right)^2-\lambda}\left(\psi_n(s)\,\psi_n(t)-\frac{2}{\pi}\sin\left(n-\tfrac{1}{2}\right)s\,\sin\left(n-\tfrac{1}{2}\right)t\right).$$

Since the results quoted in that paragraph may be shown to lead to

$$\lim_{\lambda \to -\infty} -\lambda\left[\mathrm{K}_\lambda(s, t)-{}_0\Gamma_\lambda(s, t)\right] = \lim_{\lambda \to \infty} \rho\,{}_0\Gamma_\lambda(s, t)\,\alpha(\rho, s, t) = 0,$$

we prove from this, by the method employed in § 9, that, for unequal values of $s$ and $t$, (27) converges to the limit zero, as $n$ tends to $\infty$.

Employing $\sigma_n(s)$ with the signification of the preceding paragraph, and denoting by ${}_0\mathfrak{s}_n(s)$ the sum of the first $n$ terms of the series

$$\frac{2}{\pi}\sin\tfrac{1}{2}s\int_0^\pi f(t)\sin\tfrac{1}{2}t\,dt+\frac{2}{\pi}\sin\tfrac{3}{2}s\int_0^\pi f(t)\sin\tfrac{3}{2}t\,dt$$
$$+\ldots+\frac{2}{\pi}\sin(n-\tfrac{1}{2})s\int_0^\pi f(t)\sin(n-\tfrac{1}{2})t\,dt+\ldots, \quad . \quad (29)$$

we deduce at once that, as $n$ tends to $\infty$,

$$\sigma_n(s)-{}_0\mathfrak{s}_n(s)$$

converges uniformly to zero on the whole of $(0, \pi)$. The corollaries to this result are of the same nature as those stated in § 10, the only difference is that ${}_0\mathfrak{s}_n(s)$ replaces $\mathfrak{s}_n(s)$, and (29) the series (25).

§ 12. We shall state the corresponding results when the pair of boundary conditions is ${}^\pi\mathrm{B}'$, or ${}^\pi_0\mathrm{B}'$, more briefly. In the case of the pair ${}^\pi\mathrm{B}'$ it will be found that (27) must be replaced by

$$\sum_{m=1}^{n}\left(\psi_m(s)\,\psi_m(t)-\frac{2}{\pi}\cos(m-\tfrac{1}{2})s\cos(m-\tfrac{1}{2})t\right). \quad . \quad . \quad . \quad . \quad (30)$$

Replacing the boundary conditions (28) by

$$\frac{du}{ds}=0 \quad \text{at} \quad s=0, \qquad u=0 \quad \text{at} \quad s=\pi,$$

it may be established that

$${}^\pi\Gamma_\lambda(s,t)=\sum_{n=1}\frac{1}{(n-\frac{1}{2})^2-\lambda}\,\frac{2}{\pi}\cos(n-\tfrac{1}{2})s\cos(n-\tfrac{1}{2})t\,;$$

whence, in virtue of the results quoted in III., § 19, it may be shown that (30) converges to zero, for unequal values of $s$ and $t$.* Finally, if ${}^\pi\mathfrak{s}_n(s)$ is employed to denote the sum of the first $n$ terms of the series

$$\frac{2}{\pi}\cos\tfrac{1}{2}s\int_0^\pi f(t)\cos\tfrac{1}{2}t\,dt+\frac{2}{\pi}\cos\tfrac{3}{2}s\int_0^\pi f(t)\cos\tfrac{3}{2}t\,dt$$
$$+\ldots+\frac{2}{\pi}\cos(n-\tfrac{1}{2})s\int_0^\pi f(t)\cos(n-\tfrac{1}{2})t\,dt+\ldots, \quad . \quad (31)$$

we obtain the result that, as $n$ tends to $\infty$,

$$\sigma_n(s)-{}^\pi\mathfrak{s}_n(s)$$

converges uniformly to zero in the interval $(0, \pi)$.

* This may be deduced from the corresponding result obtained in the preceding paragraph. For, if $\psi_n(s)$ is the normal function which, for $\mu=\lambda_n$, satisfies

$$\frac{d^2u}{ds^2}+(q+\mu)u=0$$

and the pair of boundary conditions ${}^\pi\mathrm{B}'$, then, for the same value of $\mu$, $\psi_n(\pi-s)$ satisfies this equation and a pair of boundary conditions of the same type as ${}_0\mathrm{B}'$.

When the pair of boundary conditions for $(0, \pi)$ is ${}_0^\pi B'$, (27) must be replaced by

$$\sum_{m=1}^{n}\left(\psi_m(s)\,\psi_m(t)-\frac{2}{\pi}\sin ms\sin mt\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The GREEN'S function of (22) for the boundary conditions

$$u=0 \quad \text{at} \quad s=0, \qquad u=0 \quad \text{at} \quad s=\pi,$$

will be found to be

$${}_0^\pi\Gamma_\lambda(s,t)=\sum_{n=1}\frac{1}{n^2-\lambda}\frac{2}{\pi}\sin ns\sin nt,$$

from which it is easily proved that, as $n$ tends to $\infty$, (32) converges to zero for unequal values of $s$ and $t$. We deduce that, if ${}_0^\pi\mathfrak{s}_n(s)$ is the sum of the first $n$ terms of the series

$$\frac{2}{\pi}\sin s\int_0^\pi f(t)\sin t\,dt+\frac{2}{\pi}\sin 2s\int_0^\pi f(t)\sin 2t\,dt+\ldots+\frac{2}{\pi}\sin ns\int_0^\pi f(t)\sin nt\,dt+\ldots, \quad . \quad (33)^*$$

then, as $n$ tends to $\infty$,

$$\sigma_n(s)-{}_0^\pi\mathfrak{s}_n(s)$$

converges uniformly to zero in the interval $(0, \pi)$.

§ 13. We proceed to investigate the behaviour of the differences between the various pairs of sums which we have denoted by $\mathfrak{s}_n(s)$, ${}_0\mathfrak{s}_n(s)$, ${}^\pi\mathfrak{s}_n(s)$, and ${}_0^\pi\mathfrak{s}_n(s)$, as $n$ tends to $\infty$. The reader will easily prove that these sums are

$$\mathfrak{s}_n(s)=\frac{1}{2\pi}\int_0^\pi f(t)\left[\frac{\sin\frac{1}{2}(2n-1)(s-t)}{\sin\frac{1}{2}(s-t)}+\frac{\sin\frac{1}{2}(2n-1)(s+t)}{\sin\frac{1}{2}(s+t)}\right]dt,$$

$${}_0\mathfrak{s}_n(s)=\frac{1}{2\pi}\int_0^\pi f(t)\left[\frac{\sin n(s-t)}{\sin\frac{1}{2}(s-t)}-\frac{\sin n(s+t)}{\sin\frac{1}{2}(s+t)}\right]dt,$$

$${}^\pi\mathfrak{s}_n(s)=\frac{1}{2\pi}\int_0^\pi f(t)\left[\frac{\sin n(s-t)}{\sin\frac{1}{2}(s-t)}+\frac{\sin n(s+t)}{\sin\frac{1}{2}(s+t)}\right]dt,$$

$${}_0^\pi\mathfrak{s}_n(s)=\frac{1}{2\pi}\int_0^\pi f(t)\left[\frac{\sin\frac{1}{2}(2n+1)(s-t)}{\sin\frac{1}{2}(s-t)}-\frac{\sin\frac{1}{2}(2n+1)(s+t)}{\sin\frac{1}{2}(s+t)}\right]dt.$$

Let us define a function $f_1(s)$ for all values of $s$ by the rules

$$f_1(s)=f(s)\quad(0\leq s\leq\pi), \qquad =0\quad(-\pi<s<0),$$
$$f_1(s+2\pi)=f_1(s);$$

and a function $f_2(s)$ by the rules

$$f_2(s)=f(s)\quad(0\leq s\leq\pi), \qquad =0\quad(-\pi<s<0),$$
$$f_2(s+2\pi)=-f_2(s).$$

* This is, of course, FOURIER'S sine series corresponding to $f(s)$.

Let $I_m(s)$ denote the value of

$$\int_0^{\pi} f(t)\frac{\sin\frac{1}{2}m(s+t)}{\sin\frac{1}{2}(s+t)}\,dt,$$

for integral values of $m$. It is easily seen that

$$I_{2n+1}(s)=\int_{-\pi}^{\pi} f_1(t)\frac{\sin\frac{1}{2}(2n+1)(s+t)}{\sin\frac{1}{2}(s+t)}\,dt.$$

Substituting $s+t=w$, and then replacing $w$ by $t$, we obtain

$$I_{2n+1}(s)=\int_{-(\pi-s)}^{\pi+s} f_1(-s+t)\frac{\sin\frac{1}{2}(2n+1)t}{\sin\frac{1}{2}t}\,dt,$$

which, owing to the periodicity of $f_1(s)$, leads to

$$I_{2n+1}(s)=\int_{-\pi}^{\pi} f_1(-s+t)\frac{\sin\frac{1}{2}(2n+1)t}{\sin\frac{1}{2}t}\,dt;$$

hence we have

$$I_{2n+1}(s)=\int_0^{\pi}[f_1(-s-t)+f_1(-s+t)]\frac{\sin\frac{1}{2}(2n+1)t}{\sin\frac{1}{2}t}\,dt.$$

In a similar way it may be proved that

$$I_{2n}(s)=\int_0^{\pi}[f_2(-s-t)+f_2(-s+t)]\frac{\sin nt}{\sin\frac{1}{2}t}\,dt. \qquad (34)$$

Let us now suppose that $s$ is a point of an interval $(\gamma, \delta)$ lying wholly within $(0, \pi)$; and let $\alpha$ be a positive number less than both $\gamma$ and $\pi-\delta$. We have

$$f_1(-s-t)=f_1(-s+t)=0 \quad (\gamma\leq s\leq\delta, \quad 0\leq t\leq\alpha).$$

Thus

$$I_{2n+1}(s)=\int_{\alpha}^{\pi}[f_1(-s-t)+f_1(-s+t)]\frac{\sin\frac{1}{2}(2n+1)t}{\sin\frac{1}{2}t}\,dt.$$

Since, by a known theorem,* the right-hand member converges uniformly to zero as $n$ tends to $\infty$, it follows that $I_{2n+1}(s)$ converges uniformly to zero in the interval $(\gamma, \delta)$. It may be shown in a similar way, by using (34), that $I_{2n}(s)$ has the same property.

§ 14. Let $I'_m(s)$ be the value of

$$\int_0^{\pi} f(t)\frac{\sin\frac{1}{2}m(s-t)}{\sin\frac{1}{2}(s-t)}\,dt.$$

* *Vide* II., § 6.

By employing the method of the preceding paragraph, and using the substitution $t-s=w$ instead of $s+t=w$, it may be shown that

$$I'_{2n+1}(s)=\int_0^\pi [f_1(s-t)+f_1(s+t)]\frac{\sin\frac{1}{2}(2n+1)t}{\sin\frac{1}{2}t}dt,$$

$$I'_{2n}(s)=\int_0^\pi [f_2(s-t)+f_2(s+t)]\frac{\sin nt}{\sin\frac{1}{2}t}dt.$$

With the same convention as to the values of $s$ and $\alpha$, it is easily shown that

$$\left.\begin{aligned}f_1(s-t)&=f_2(s-t)=f(s-t)\\ f_1(s+t)&=f_2(s+t)=f(s+t).\end{aligned}\right\}\quad(\gamma\leq s\leq\delta,\ 0\leq t\leq\alpha)$$

Hence

$$I'_{2n+1}(s)-I'_{2n}(s)=\int_0^\alpha [f(s-t)+f(s+t)]\frac{\sin\frac{1}{2}(2n+1)t-\sin nt}{\sin\frac{1}{2}t}dt$$
$$+\int_\alpha^\pi [f_1(s-t)+f_1(s+t)]\frac{\sin\frac{1}{2}(2n+1)t}{\sin\frac{1}{2}t}dt-\int_\alpha^\pi [f_2(s-t)+f_2(s+t)]\frac{\sin nt}{\sin\frac{1}{2}t}dt.$$

The first integral on the right-hand side of this equation is

$$\int_0^\alpha [f(s-t)+f(s+t)]\frac{\cos(n+\frac{1}{4})t}{\cos\frac{1}{4}t}dt,$$

which may be written

$$\int_0^\alpha [f(s-t)+f(s+t)]\cos nt\,dt-\int_0^\alpha [f(s-t)+f(s+t)]\tan\tfrac{1}{4}t\sin nt\,dt;$$

hence, applying the first corollary of II., § 9, we see that it converges uniformly to zero, for values of $s$ in $(\gamma,\delta)$. Since the two other integrals have this property in virtue of the same corollary, we conclude that, as $n$ tends to $\infty$,

$$I'_{2n+1}(s)-I'_{2n}(s)$$

converges uniformly to zero, for values of $s$ in $(\gamma,\delta)$. As a corollary we deduce that

$$I'_{2n+2}(s)-I'_{2n}(s)$$

has the same property.

§ 15. Referring to the formulæ of § 13, it is evident that

$$\mathfrak{s}_n(s)=\frac{1}{2\pi}[I'_{2n-1}(s)+I_{2n-1}(s)],\quad {}_0\mathfrak{s}_n(s)=\frac{1}{2\pi}[I'_{2n}(s)-I_{2n}(s)].$$

Hence

$$\mathfrak{s}_n(s)-{}_0\mathfrak{s}_n(s)=\frac{1}{2\pi}\{[I'_{2n-1}(s)-I'_{2n}(s)]+I_{2n-1}(s)+I_{2n}(s)\}.$$

As each of the three terms within the bracket has been shown to converge uniformly to zero in $(\gamma, \delta)$, it follows that, *as n tends to* $\infty$, the difference

$$\mathfrak{s}_n(s) - {}_0\mathfrak{s}_n(s)$$

converges uniformly to zero for values of $s$ in $(\gamma, \delta)$. It may be shown in the same way that each of the differences

$$\mathfrak{s}_n(s) - {}^{\pi}\mathfrak{s}_n(s), \qquad \mathfrak{s}_n(s) - {}_0^{\pi}\mathfrak{s}_n(s),$$

has the same property.

Since $(\gamma, \delta)$ is any interval lying within $(0, \pi)$, it follows that, as $n$ tends to $\infty$, the difference between any two of the sums $\mathfrak{s}_n(s)$, ${}_0\mathfrak{s}_n(s)$, ${}^{\pi}\mathfrak{s}_n(s)$, ${}_0^{\pi}\mathfrak{s}_n(s)$, converges to zero at each point of the *open* interval $(0, \pi)$. It remains to consider the end points of the interval. At $s = 0$ we have

$$\mathfrak{s}_n(0) = \frac{1}{\pi}\int_0^{\pi} f(t)\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt, \qquad {}^{\pi}\mathfrak{s}_n(0) = \frac{1}{\pi}\int_0^{\pi} f(t)\frac{\sin nt}{\sin\frac{1}{2}t}\,dt,$$

and, of course,

$${}_0\mathfrak{s}_n(0) = {}_0^{\pi}\mathfrak{s}_n(0) = 0.$$

From the first two formulæ we obtain

$$\begin{aligned}{}^{\pi}\mathfrak{s}_n(0) - \mathfrak{s}_n(0) &= \frac{1}{\pi}\int_0^{\pi} f(t)\frac{\sin nt - \sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt \\ &= \frac{1}{\pi}\int_0^{\pi} f(t)\cos nt\,dt + \frac{1}{\pi}\int_0^{\pi} f(t)\tan\tfrac{1}{4}t\sin nt\,dt.\end{aligned}$$

As both $f(t)$ and $f(t)\tan\frac{1}{4}t$ possess Lebesgue integrals in $(0, \pi)$, the integrals on the right converge to zero, as $n$ tends to $\infty$.* We thus see that

$$\lim_{n\to\infty}\left[{}^{\pi}\mathfrak{s}_n(0) - \mathfrak{s}_n(0)\right] = 0.$$

In a similar way it may be shown that

$$\lim_{n\to\infty}\left[{}_0\mathfrak{s}_n(\pi) - \mathfrak{s}_n(\pi)\right] = 0,$$

whilst

$${}^{\pi}\mathfrak{s}_n(\pi) = {}_0^{\pi}\mathfrak{s}_n(\pi) = 0,$$

for all values of $n$.

After what was said in the corresponding case dealt with in § 10, the reader will perceive the bearing of the results of this paragraph upon the convergence of the four trigonometric series (25), (29), (31), and (33).

§ 16. The reader is now asked to review the results which have been obtained above. It was shown in §§ 10–12 that the limits of indeterminacy of any canonical

* II., § 6.

Sturm-Liouville series corresponding to $f(s)$ are identical with those of one of the four series

$$\frac{1}{\pi}\int_0^\pi f(t)\,dt+\frac{2}{\pi}\cos s\int_0^\pi f(t)\cos t\,dt+\ldots+\frac{2}{\pi}\cos(n-1)s\int_0^\pi f(t)\cos(n-1)t\,dt+\ldots, \quad (25)$$

$$\frac{2}{\pi}\sin\tfrac{1}{2}s\int_0^\pi f(t)\sin\tfrac{1}{2}t\,dt+\frac{2}{\pi}\sin\tfrac{3}{2}s\int_0^\pi f(t)\sin\tfrac{3}{2}t\,dt$$
$$+\ldots+\frac{2}{\pi}\sin(n-\tfrac{1}{2})s\int_0^\pi f(t)\sin(n-\tfrac{1}{2})t\,dt+\ldots, \quad (29)$$

$$\frac{2}{\pi}\cos\tfrac{1}{2}s\int_0^\pi f(t)\cos\tfrac{1}{2}t\,dt+\frac{2}{\pi}\cos\tfrac{3}{2}s\int_0^\pi f(t)\cos\tfrac{3}{2}t\,dt$$
$$+\ldots+\frac{2}{\pi}\cos(n-\tfrac{1}{2})s\int_0^\pi f(t)\cos(n-\tfrac{1}{2})t\,dt+\ldots, \quad (31)$$

$$\frac{2}{\pi}\sin s\int_0^\pi f(t)\sin t\,dt+\frac{2}{\pi}\sin 2s\int_0^\pi f(t)\sin 2t\,dt+\ldots+\frac{2}{\pi}\sin ns\int_0^\pi f(t)\sin nt\,dt+\ldots, \quad (33)$$

at each point of the closed interval $(0, \pi)$. It was then shown in §§ 13–15 that, at each point of the open interval $(0, \pi)$, each of these four series has the same limits of indeterminacy. We have therefore established the following theorem :—

I. *At any point of the open interval $(0, \pi)$ each of the canonical Sturm-Liouville series corresponding to an assigned function which is integrable in $(0, \pi)$ in accordance with* LEBESGUE'S *definition has the same limits of indeterminacy.*

In particular we have :—

II. *If any one of the canonical Sturm-Liouville series corresponding to the function converges at a point of the open interval $(0, \pi)$, then all of them converge at this point, and all have the same sum.*

It was shown in §§ 10, 12, that, at the end point $s = 0$, all canonical Sturm-Liouville series whose normal functions satisfy B′ have the same limits of indeterminacy as (25), and that those whose normal functions satisfy $^{\pi}$B′ have the same limits of indeterminacy as (31) at this point. Then, in § 15, we proved that (25) and (31) have the same limits of indeterminacy at $s = 0$. Since similar remarks apply to the end point $s = \pi$, we have the theorem :—

III. *All those canonical Sturm-Liouville series corresponding to the function, whose normal functions do not satisfy the boundary condition $u = 0$ at $\begin{matrix} s=0 \\ s=\pi \end{matrix}$, have the same limits of indeterminacy at $\begin{matrix} s=0 \\ s=\pi \end{matrix}$.*

In particular, the reader will observe that, if one of the series mentioned converges at an end point, all do so.

Lastly, if the reader will examine the results obtained in §§ 10–15, he will find that we have established the theorem :—

IV. *If any one of the canonical Sturm-Liouville series corresponding to the* *assigned function converges uniformly in a set of points which, together with its limiting points, is contained within* $(0, \pi)$, *then all of these series converge uniformly in the set.*

The results obtained in §§ 10–13 show that, if any one of the canonical Sturm-Liouville series corresponding to the function converges uniformly in *any* set of points of $(0, \pi)$, then all of the series whose normal functions satisfy a pair of boundary conditions of the same category ($B'$, ${}_0B'$, ${}^{\pi}B'$, or ${}_0^{\pi}B'$) as the first-mentioned series converge uniformly in the set.

§ 17. As usual, let $f(s)$ be any function which has a Lebesgue integral in $(0, \pi)$. From theorem I of the preceding paragraph it appears that, at a point $s$ of the open interval $(0, \pi)$, all canonical Sturm-Liouville series corresponding to $f(s)$ have the same limits of indeterminacy as FOURIER's cosine series corresponding to $f(s)$; these limits are therefore

$$\overline{\lim_{n\to\infty}}\, \mathfrak{s}_n(s).$$

From the results obtained in §§ 13, 14, it is evident that they are

$$\overline{\lim_{n\to\infty}} \frac{1}{2\pi}\int_0^{\pi}[f_1(s-t)+f_1(s+t)]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt.*$$

Now, if $\alpha$ is any positive number less than $\pi$, we have

$$\lim_{n\to\infty}\int_{\alpha}^{\pi} f_1(s\pm t)\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt = 0,$$

by the last corollary of II., § 6. It follows that, at a point $s$ of the open interval $(0, \pi)$, the limits of indeterminacy of the canonical Sturm-Liouville series corresponding to $f(s)$ are

$$\overline{\lim_{n\to\infty}} \frac{1}{2\pi}\int_0^{\alpha}[f_1(s-t)+f_1(s+t)]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt.$$

Recalling that $\alpha$ is arbitrarily small, we have thus established the theorem :—†

*At any particular point of the open interval* $(0, \pi)$ *the limits of indeterminacy of the canonical Sturm-Liouville series corresponding to an assigned function depend only upon the values assumed by the function in an arbitrarily small neighbourhood of the point.*

Let us next consider the case when $s$ is an end point of $(0, \pi)$. It follows from theorem III that the limits of indeterminacy of those canonical Sturm-Liouville series whose normal functions do not all vanish at an end point are the values of

$$\overline{\lim_{n\to\infty}}\, \mathfrak{s}_n(s)$$

* Since $\lim_{n\to\infty} I_{2n-1}(s) = 0$.

† For the theorems of this paragraph *cf.* HOBSON's paper cited in the Introduction. HOBSON, it will be recalled, assumes that $q$ has limited total fluctuation in $(0, \pi)$.

at this point. It may be proved, from the formula

$$s_n(0) = \frac{1}{\pi}\int_0^\pi f(t)\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt,$$

that, at $s = 0$, these limits are

$$\overline{\underline{\lim}}_{n\to\infty}\frac{1}{\pi}\int_0^\alpha f(t)\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt\,;$$

and it will be found that, at $s = \pi$, they are

$$\overline{\underline{\lim}}_{n\to\infty}\frac{1}{\pi}\int_0^\alpha f(\pi-t)\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt\,;$$

the numbers $\alpha$ being positive and arbitrarily small in each case. Hence we have the theorem :—

*At an end point of* $(0, \pi)$ *the limits of indeterminacy of all canonical Sturm-Liouville series corresponding to an assigned function, save those whose normal functions satisfy the boundary condition* $u = 0$ *there, depend only upon the values assumed by the function in an arbitrarily small neighbourhood to the right, or left, of the point, as the case may be.*

Again, from theorem IV, it appears that, if one of the Fourier's series, say the cosine series, corresponding to $f(s)$ converges uniformly in an interval $(\gamma, \delta)$ contained within $(0, \pi)$, then all canonical Sturm-Liouville series corresponding to $f(s)$ converge uniformly in $(\gamma, \delta)$. It was shown in § 13 that

$$s_n(s) - \frac{1}{2\pi}\int_0^\pi [f_1(s-t) + f_1(s+t)]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt^*$$

converges uniformly to zero for values of $s$ in $(\gamma, \delta)$, as $n$ increases indefinitely. Since both the integrals

$$\int_\alpha^\pi f_1(s\pm t)\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt$$

converge uniformly to zero, for these values of $s$,† and since

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^\alpha \frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt = 1,\ddagger \qquad (35)$$

* This is clearly $\frac{1}{2\pi}\,I_{2n-1}(s)$.

† In virtue of the first corollary of II, § 9.

‡ This result follows from the fact that

$$\frac{1}{\pi}\int_0^\pi \frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt = \frac{1}{\pi}\int_0^\pi [1 + 2\sum_{r=1}^{n-1}\cos rt]\,dt = 1,$$

and that

$$\lim_{n\to\infty}\frac{1}{\pi}\int_\alpha^\pi \frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt = 0,$$

by the last corollary of II., § 6.

it is thus clear that the canonical Sturm-Liouville series will all converge uniformly to $f(s)$ in $(\gamma, \delta)$, when

$$\int_0^\alpha [f_1(s-t)+f_1(s+t)-2f(s)]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt$$

converges uniformly to zero in this interval. We have thus proved the theorem :—

*The answer to the question whether the canonical Sturm-Liouville series corresponding to an assigned function all converge uniformly, or not, in an interval* $(\gamma, \delta)$ *contained within* $(0, \pi)$ *depends only upon the values assumed by the function in an interval enclosing* $(\gamma, \delta)$ *in its interior, and exceeding it in length by an arbitrarily small amount.*

It is evidently a necessary condition for uniform convergence that the function should be continuous in $(\gamma, \delta)$.

§ 18. After what was said in the preceding paragraph, it will be clear that, when $\lim\limits_{n\to\infty} s_n(s)$ exists as a finite number at a point of the open interval $(0, \pi)$, all the canonical Sturm-Liouville series corresponding to $f(s)$ converge at this point, and have the value of this limit for their common sum. We may employ this fact to obtain conditions under which these series converge.

Let $\omega(s)$, the bilateral limit of $f(s)$ at the point $s$, exist as a finite number. In virtue of (35), it is easily seen that

$$s_n(s)-\omega(s) \underset{n\to\infty}{=\!=} \frac{1}{2\pi}\int_0^\alpha [f_1(s-t)+f_1(s+t)-2\omega(s)]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt. \quad . \quad (36)$$

Let us now suppose that, as $\beta$ diminishes indefinitely, the integral

$$\int_0^\beta [f_1(s-t)+f_1(s+t)-2\omega(s)]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt$$

converges uniformly to zero for positive integral values of $n$. When any positive number $\epsilon$ is assigned, we may then choose $\alpha$ in such a way that the numerical value of the right-hand member of (36) is less than $\frac{1}{2}\epsilon$ for these values of $n$. Further, $\alpha$ being fixed, a positive integer N may be chosen great enough to ensure that the difference between the left- and right-hand members is numerically less than $\frac{1}{2}\epsilon$, for $n \geq$ N. Thus we have

$$|s_n(s)-\omega(s)| < \epsilon,$$

for $\geq$ N ; and therefore

$$\lim_{n\to\infty} s_n(s) = \omega(s).$$

We have now established the theorem :—

*At a point s of the open interval* $(0, \pi)$, *where* $\omega(s)$, *the bilateral limit of* $f(s)$, *exists as a finite number, a sufficient condition that all the canonical Sturm-Liouville* *series corresponding to* $f(s)$ *may converge, and therefore have* $\omega(s)$ *for their common sum, is that, as* $\beta$ *diminishes indefinitely,*

$$\int_0^\beta [f_1(s-t)+f_1(s+t)-2\omega(s)] \frac{\sin \frac{1}{2}(2n-1)t}{\sin \frac{1}{2}t} dt$$

*should converge uniformly to zero, for positive integral values of* $n$.

We proceed to verify that this condition is satisfied when $f(s)$ has limited total fluctuation in an arbitrarily small neighbourhood $(s-\alpha, s+\alpha)$ of the point $s$. Since a function which has limited total fluctuation is the difference of two monotone functions, we may confine ourselves to the case in which $f(s)$ is monotone in the neighbourhood.

Observing that

$$2\omega(s) = f(s-0)+f(s+0),$$

and that, for sufficiently small values of $t$, $f_1(s\pm t)$ may be replaced by $f(s\pm t)$, it is plainly sufficient to show that each of the integrals

$$\int_0^\beta [f(s-t)-f(s-0)] \frac{\sin \frac{1}{2}(2n-1)t}{\sin \frac{1}{2}t} dt, \quad \int_0^\beta [f(s+t)-f(s+0)] \frac{\sin \frac{1}{2}(2n-1)t}{\sin \frac{1}{2}t} dt$$

converges uniformly to zero, as $\beta$ diminishes indefinitely.

For $\beta \leq \alpha$, the first integral is

$$[f(s-\beta)-f(s-0)] \int_{\beta_1}^\beta \frac{\sin \frac{1}{2}(2n-1)t}{\sin \frac{1}{2}t} dt,$$

where, in accordance with the second mean value theorem, $0 \leq \beta_1 \leq \beta$. It will therefore have this property, if

$$\int_{\beta_1}^\beta \frac{\sin \frac{1}{2}(2n-1)t}{\sin \frac{1}{2}t} dt$$

is limited, for values of $\beta$ and $\beta_1$ in $(0, \alpha)$, and for positive integral values of $n$. Now we have

$$\frac{\sin \frac{1}{2}(2n-1)t}{\sin \frac{1}{2}t} = 1+2\sum_{m=1}^{n-1} \cos mt.$$

Hence, by integration,

$$\int_{\beta_1}^\beta \frac{\sin \frac{1}{2}(2n-1)t}{\sin \frac{1}{2}t} dt = \beta-\beta_1+2\sum_{m=1}^{n-1} \frac{\sin m\beta}{m} - 2\sum_{m=1}^{n-1} \frac{\sin m\beta_1}{m},$$

from which it is evident that the left-hand member is limited for the stated values of $\beta$, $\beta_1$, and $n$. As similar remarks apply to the second integral, we have the theorem :—*

*If* $f(s)$ *has limited total fluctuation in an arbitrarily small neighbourhood of a*

* Obtained by HOBSON for the case in which $q$ has limited total fluctuation in $(0, \pi)$, *vide* the paper cited in the Introduction. The same remark applies to the corresponding theorems of §§ 19, 20.

*point* $(s)$ *belonging to the open interval* $(0, \pi)$, *then all the canonical Sturm-Liouville series corresponding to* $f(s)$ *converge, and have the sum* $\frac{1}{2}[f(s-0)+f(s+0)]$, *at this point.*

Again, it is easy to see that the above condition is satisfied when

$$\left|\frac{f_1(s-t)+f_1(s+t)-2\omega(s)}{t}\right|$$

has a Lebesgue integral with respect to $t$ in an interval $(0, \alpha)$. For, in this case,

$$\left|\frac{f_1(s-t)+f_1(s+t)-2\omega(s)}{\sin\frac{1}{2}t}\right|$$

also has a Lebesgue integral in the interval, so that

$$\left|\int_0^\beta [f_1(s-t)+f_1(s+t)-2\omega(s)]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt\right| \leqq \int_0^\beta \left|\frac{f_1(s-t)+f_1(s+t)-2\omega(s)}{\sin\frac{1}{2}t}\right|dt,$$

where, by a known property of Lebesgue integrals, the right-hand member tends to zero with $\beta$.

Since $f_1(s\pm t)$ may be replaced by $f(s\pm t)$ for values of $\alpha$ which are sufficiently small, we have thus established the theorem :—

*At any point* $s$ *of the open interval* $(0, \pi)$, *where* $\omega(s)$, *the bilateral limit of* $f(s)$, *exists as a finite number, a sufficient condition that all the canonical Sturm-Liouville series corresponding to* $f(s)$ *may converge, and therefore have* $\omega(s)$ *for their common sum, is that*

$$\left|\frac{f(s-t)+f(s+t)-2\omega(s)}{t}\right|$$

*should have a Lebesgue integral with respect to* $t$ *in an interval* $(0, \alpha)$.

In particular we have the following corollary :—

*At any point* $s$ *of the open interval* $(0, \pi)$, *where* $f(s-0)$ *and* $f(s+0)$ *exist as finite numbers, a sufficient condition that all the canonical Sturm-Liouville series corresponding to* $f(s)$ *may converge, and therefore have* $\frac{1}{2}[f(s-0)+f(s+0)]$ *for their common sum, is that*

$$\left|\frac{f(s-t)-f(s-0)}{t}\right|, \qquad \left|\frac{f(s+t)-f(s+0)}{t}\right|$$

*should both possess Lebesgue integrals in an interval* $(0, \alpha)$.*

§ 19. Consider next the case in which $s$ has the value zero. It follows from what was said in § 17 that, with the exception of those whose normal functions satisfy the boundary condition $u = 0$ at this point, all canonical Sturm-Liouville series

* For an amplification of these theorems *cf.* HOBSON, 'The Theory of Functions of a Real Variable,' pp. 680–683.

corresponding to $f(s)$ will converge at $s=0$, and have $\lim_{n\to\infty} s_n(0)$ for their common sum, provided that this limit exists and is finite.

Let us suppose that $f(0+0)$ exists and is finite. Then it is easily seen that

$$s_n(0)-f(0+0) \underset{n\to\infty}{=\!=} \frac{1}{\pi}\int_0^{\alpha}[f(t)-f(0+0)]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt.$$

It follows from this, by a proof similar to that employed above, that the series mentioned will all converge to $f(0+0)$, if, as $\beta$ diminishes indefinitely, the integral

$$\int_0^{\beta}[f(t)-f(0+0)]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt$$

converges uniformly to zero, for positive integral values of $n$. The reader will be able to show that this condition is satisfied when $f(s)$ has limited total fluctuation in an arbitrarily small neighbourhood to the right of $s=0$; and also when

$$\left|\frac{f(t)-f(0+0)}{t}\right|$$

possesses a Lebesgue integral in an interval $(0,\alpha)$.

As corresponding remarks apply at the end point $s=\pi$, we have the theorems:—

*If $f(s)$ has limited total fluctuation in an arbitrarily small neighbourhood to the $\begin{matrix}\text{right of } s=0\\ \text{left of } s=\pi\end{matrix}$, then those canonical Sturm-Liouville series corresponding to $f(s)$, whose normal functions do not satisfy the boundary condition $u=0$ at $\begin{matrix}s=0\\ s=\pi\end{matrix}$, converge and have the sum $\begin{matrix}f(0+0)\\ f(\pi-0)\end{matrix}$ at this point.*

*If $\begin{matrix}f(0+0)\\ f(\pi-0)\end{matrix}$ exists as a finite number, and $\begin{matrix}\left|\dfrac{f(t)-f(0+0)}{t}\right|\\ \left|\dfrac{f(\pi-t)-f(\pi-0)}{t}\right|\end{matrix}$ has a Lebesgue integral with respect to $t$ in an interval $(0,\alpha)$, then those canonical Sturm-Liouville series corresponding to $f(s)$, whose normal functions do not satisfy the boundary condition $u=0$ at $\begin{matrix}s=0\\ s=\pi\end{matrix}$, converge and have the sum $\begin{matrix}f(0+0)\\ f(\pi-0)\end{matrix}$ at this point.*

§ 20. We saw in § 17 that all canonical Sturm-Liouville series corresponding to $f(s)$ will converge uniformly in $(\gamma,\delta)$, if

$$\int_0^{\alpha}[f_1(s-t)+f_1(s+t)-2f(s)]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt$$

converges uniformly to zero in this interval. After what was said in § 18, the reader





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will have no difficulty in showing that a sufficient condition for this is that, as $\beta$ diminishes indefinitely,

$$\int_0^\beta [f_1(s-t)+f_1(s+t)-2f(s)]\,\frac{\sin\frac{1}{2}(2n-1)\,t}{\sin\frac{1}{2}t}\,dt$$

should converge uniformly to zero, for values of $s$ in $(\gamma, \delta)$, and for positive integral values of $n$.

Let us suppose that the closed interval $(\gamma, \delta)$ belongs to the set of points of $(0, \pi)$ at which $f(s)$ is continuous. It may be shown that the sufficient condition just stated is satisfied when $f(s)$ has limited total fluctuation in an interval $(\gamma', \delta')$ enclosing $(\gamma, \delta)$ in its interior. In doing so, we may evidently confine ourselves to the case in which $f(s)$ is monotone; for, in the most general case, $f(s)$ is the difference of two functions, each of which is monotone in $(\gamma', \delta')$, and has the points of $(\gamma, \delta)$ for points of continuity.

For values of $\beta$ which are not greater than a certain positive number $\bar{\beta}$, we have

$$\int_0^\beta [f(s-t)-f(s)]\,\frac{\sin\frac{1}{2}(2n-1)\,t}{\sin\frac{1}{2}t}\,dt = [f(s-\beta)-f(s)]\int_{\beta_1}^\beta \frac{\sin\frac{1}{2}(2n-1)\,t}{\sin\frac{1}{2}t}\,dt \quad (37)$$

at each point of $(\gamma, \delta)$, where, as in § 18, we have $0 \leqq \beta_1 \leqq \beta$. The integral on the right has been shown to be limited, for values of $\beta$ and $\beta_1$ in $(0, \bar{\beta})$, and for all positive integral values of $n$. Further, as $\beta$ diminishes indefinitely, $f(s-\beta)-f(s)$ converges uniformly to zero, for values of $s$ in $(\gamma, \delta)$; this follows from our hypothesis as to the continuity of $f(s)$. We conclude, therefore, that the integral on the left of (37) converges to zero, as $\beta$ diminishes indefinitely, uniformly for values of $s$ in $(\gamma, \delta)$, and for positive integral values of $n$. As the integral

$$\int_0^\beta [f(s+t)-f(s)]\,\frac{\sin\frac{1}{2}(2n-1)\,t}{\sin\frac{1}{2}t}\,dt$$

may be shown to have the same property, it will be now clear that we have established the theorem :—

*If the set of points at which $f(s)$ is continuous includes a closed interval $(\gamma, \delta)$ lying within $(0, \pi)$, and if $f(s)$ has limited total fluctuation in an interval $(\gamma', \delta')$ enclosing $(\gamma, \delta)$ in its interior, then all the canonical Sturm-Liouville series corresponding to $f(s)$ converge uniformly in $(\gamma, \delta)$.*

Again, let us suppose that, for each value of $s$ in $(\gamma, \delta)$,

$$\left|\frac{f_1(s-t)+f_1(s+t)-2f(s)}{t}\right|$$

has a Lebesgue integral with respect to $t$ in an interval $(0, \beta)$. Then clearly

$$\left|\int_0^\beta [f_1(s-t)+f_1(s+t)-2f(s)]\frac{\sin\frac{1}{2}(2n-1)\,t}{\sin\frac{1}{2}t}\,dt\right| \leqq \frac{\beta}{\sin\frac{1}{2}\beta}\int_0^\beta \left|\frac{f_1(s-t)+f_1(s+t)-2f(s)}{t}\right| dt.$$

We at once deduce the theorem :—

*A sufficient condition that all the canonical Sturm-Liouville series corresponding to $f(s)$ may converge uniformly in an interval $(\gamma, \delta)$ lying within $(0, \pi)$ is that*

$$\int_0^\beta \left| \frac{f(s-t)+f(s+t)-2f(s)}{t} \right| dt$$

*should exist as a Lebesgue integral for each value of $s$ in $(\gamma, \delta)$, and that as $\beta$ diminishes indefinitely, it should converge uniformly to zero in $(\gamma, \delta)$. In particular, it is sufficient that*

$$\int_0^\beta \left| \frac{f(s-t)-f(s)}{t} \right| dt, \qquad \int_0^\beta \left| \frac{f(s+t)-f(s)}{t} \right| dt$$

*should both exist, and converge uniformly to zero.**

§ 21. Let $s$ be a point belonging to any interval $(\gamma, \delta)$ which is contained within $(0, \pi)$. Let $\sigma_n(s)$ be the sum of the first $n$ terms of any canonical Sturm-Liouville series at the point $s$, and let $\mathfrak{s}_n(s)$ have the signification of § 10. It follows from the results obtained in §§ 10–15 that, as $n$ increases indefinitely,

$$\sigma_n(s) - \mathfrak{s}_n(s)$$

converges uniformly to zero in $(\gamma, \delta)$. Hence, in virtue of the lemma of II., § 10, we see that

$$\frac{\sigma_1(s)+\sigma_2(s)+\ldots+\sigma_n(s)}{n} - \frac{\mathfrak{s}_1(s)+\mathfrak{s}_2(s)+\ldots+\mathfrak{s}_n(s)}{n}$$

converges uniformly to zero in this interval. In particular, since $(\gamma, \delta)$ is any interval contained within $(0, \pi)$, we see that this difference converges to zero at each point of the open interval $(0, \pi)$. It will be seen from the results of §§ 10–15 that this is also true at an end point, provided that the normal functions of the Sturm-Liouville series do not all vanish there.

Now we have (§§ 13, 14)

$$\mathfrak{s}_n(s) = \frac{1}{2\pi}\{\mathrm{I}_{2n-1}(s)+\mathrm{I}'_{2n-1}(s)\} = \frac{1}{\pi}\int_0^\pi \mathrm{F}(t)\,\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt,$$

where $\mathrm{F}(t) = \frac{1}{2}[f_1(-s-t)+f_1(-s+t)+f_1(s-t)+f_1(s+t)]$. Hence we obtain

$$\frac{\mathfrak{s}_1(s)+\mathfrak{s}_2(s)+\ldots+\mathfrak{s}_n(s)}{n} = \frac{1}{n\pi}\int_0^\pi \mathrm{F}(t)\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2 dt.$$

* It is well known that, as $n$ increases indefinitely, the integral

$$\int_0^\alpha [f_1(s-t)+f_1(s+t)-2f(s)]\,\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt$$

converges uniformly to zero for values of $s$ in $(\gamma, \delta)$, if, as $t$ diminishes indefinitely,

$$[f(s-t)+f(s+t)-2f(s)]\log t$$

converges uniformly to zero in an interval containing $(\gamma, \delta)$ in its interior (*vide* HOBSON, 'The Theory of Functions of a Real Variable,' pp. 691–694). This gives another condition for the uniform convergence of Sturm-Liouville series. The formal statement of it is left to the reader.

Since each of the four functions

$$\frac{1}{n}\int_{\alpha}^{\pi} f_1(\pm s \pm t)\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2 dt \quad (0 < \alpha < \pi)$$

converges uniformly to zero in any finite interval* (II., § 10), we see from this that, for any positive value of $\alpha$ less than $\pi$,

$$\frac{\mathfrak{s}_1(s)+\mathfrak{s}_2(s)+\ldots+\mathfrak{s}_n(s)}{n} - \frac{1}{n\pi}\int_0^{\alpha} \mathrm{F}(t)\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2 dt \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

converges uniformly to zero in $(0, \pi)$, as the positive integer $n$ increases indefinitely.

Let us suppose that $s$ is a fixed point of the closed interval $(0, \pi)$ at which $\overline{\mathrm{F}(+0)}$ has a finite value. When any positive number $\epsilon$ is assigned, we may then choose $\alpha$ small enough to ensure that

$$\mathrm{F}(t) \leq \overline{\mathrm{F}(+0)}+\epsilon,$$

at all points of $(0, \alpha)$. With this choice of $\alpha$, we have

$$\frac{1}{n\pi}\int_0^{\alpha} \mathrm{F}(t)\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2 dt \leq [\overline{\mathrm{F}(+0)}+\epsilon]\,\frac{1}{n\pi}\int_0^{\alpha}\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2 dt,$$

which, since

$$\lim_{n\to\infty}\frac{1}{n\pi}\int_0^{\alpha}\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2 dt = 1\dagger$$

and $\epsilon$ is arbitrarily small, leads to

$$\overline{\lim_{n\to\infty}}\,\frac{\mathfrak{s}_1(s)+\mathfrak{s}_2(s)+\ldots+\mathfrak{s}_n(s)}{n} \leq \overline{\mathrm{F}(+0)}.$$

The reader will have no difficulty in seeing that this inequality is valid when $\overline{\mathrm{F}(+0)}$ has one of the improper values $\pm\infty$ (*cf.* III., § 13). It may be proved in the same way that

$$\underline{\lim_{n\to\infty}}\,\frac{\mathfrak{s}_1(s)+\mathfrak{s}_2(s)+\ldots+\mathfrak{s}_n(s)}{n} \geq \underline{\mathrm{F}(+0)}.$$

From these inequalities it is evident that, at any point of $(0, \pi)$ where $\mathrm{F}(+0)$ exists,

$$\lim_{n\to\infty}\frac{\mathfrak{s}_1(s)+\mathfrak{s}_2(s)+\ldots+\mathfrak{s}_n(s)}{n}$$

exists and is equal to it. Moreover, in virtue of our definition of $f_1(s)$,‡ it will be seen that at a point of the open interval $\mathrm{F}(+0)$ is $\lim\limits_{t\to 0}\frac{1}{2}[f(s-t)+f(s+t)]$; that at $s = 0$ it is $f(0+0)$; and that at $s = \pi$ it is $f(\pi-0)$.

* This seems to have escaped notice hitherto.

† The simplest method of obtaining this result is to apply CAUCHY's theorem to (35) above.

‡ § 13.

Again, let us suppose that the set of points at which $f(s)$ is continuous includes a closed interval $(\gamma, \delta)$ lying within $(0, \pi)$. Then, supposing $\alpha < \gamma$ and $< \pi-\delta$, we have

$$F(t) = \tfrac{1}{2}[f(s-t)+f(s+t)] \quad (0 \leq t \leq \alpha)\,(\gamma \leq s \leq \delta).$$

It follows from our hypothesis as to the continuity of $f(s)$ that, when any positive number $\epsilon$ is assigned, we may choose $\alpha$ so small that

$$|F(t)-f(s)| < \tfrac{1}{3}\epsilon,$$

for these values of $s$ and $t$. Hence, observing that

$$\frac{1}{n\pi}\int_0^\pi \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt = 1$$

we obtain

$$\left|\frac{1}{n\pi}\int_0^\alpha F(t)\left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt - f(s)\right| < \frac{|f(s)|}{n\pi}\int_\alpha^\pi \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt + \frac{\epsilon}{3n\pi}\int_0^\alpha \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 dt.$$

The second term on the right is not greater than $\frac{1}{3}\epsilon$, and, since

$$\lim_{n\to\infty} \frac{1}{n}\int_\alpha^\pi \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2 = 0,$$

we can clearly find a positive integer $N_1$ great enough to ensure that the first is not greater than $\frac{1}{3}\epsilon$, for all values of $n \geq N_1$, and of $s$ in $(\gamma, \delta)$. Again, we can choose a positive integer $N_2$ in such a way that the numerical value of the difference (38) is less than $\frac{1}{3}\epsilon$, for all values of $n \geq N_2$, and of $s$ in $(\gamma, \delta)$. It follows at once that, for values of $n$ which are not less than the greatest of $N_1$ and $N_2$, we have

$$\left|\frac{\varsigma_1(s)+\varsigma_2(s)+\ldots+\varsigma_n(s)}{n} - f(s)\right| < \epsilon$$

for all values of $s$ in $(\gamma, \delta)$. In other words, we have shown that, as $n$ increases indefinitely,

$$\frac{\varsigma_1(s)+\varsigma_2(s)+\ldots+\varsigma_n(s)}{n}$$

converges uniformly to $f(s)$ for these values of $s$.

We may summarise the results obtained in this paragraph in the following theorems :—

*The arithmetic mean of the first $n$ partial sums of any canonical Sturm-Liouville series corresponding to $f(s)$ converges to $\lim_{t\to 0} \frac{1}{2}[f(s-t)+f(s+t)]$, as $n$ increases indefinitely, at each point of the open interval $(0, \pi)$ where this limit exists as a finite number; moreover, at a point where this limit has one of the improper values $\pm\infty$, it diverges to this value and is non-oscillatory. If the set of points at which $f(s)$ is* *continuous includes a closed interval lying within* $(0, \pi)$, *then the arithmetic mean converges to* $f(s)$ *uniformly in this interval.*

*If the normal functions of a Sturm-Liouville series do not satisfy the boundary condition* $u = 0$ *at* $\begin{matrix} s = 0 \\ s = \pi \end{matrix}$, *then, as n increases indefinitely, the arithmetic mean of the first n partial sums of the series converges at* $\begin{matrix} s = 0 \\ s = \pi \end{matrix}$ *to* $\begin{matrix} f(0+0) \\ f(\pi-0) \end{matrix}$, *whenever this limit exists as a finite number; moreover, when this limit has one of the improper values* $\pm\infty$, *the arithmetic mean diverges to this value and is non-oscillatory.*

§ 22. We proceed to apply the foregoing results to the more general Sturm-Liouville series

$$\Psi_1(x)\int_a^b g(y)\,\Psi_1(y)\,\mathrm{F}(y)\,dy + \Psi_2(x)\int_a^b g(y)\,\Psi_2(y)\,\mathrm{F}(y)\,dy$$
$$+\ldots+\Psi_n(x)\int_a^b g(y)\,\Psi_n(y)\,\mathrm{F}(y)\,dy+\ldots \quad . \quad . \quad (39)$$

We saw above (III., § 22) that the terms of this series are identical with those of

$$\frac{\psi_1(s)}{w(s)}\int_0^\pi \psi_1(t)\,w(t)f(t)\,dt + \frac{\psi_2(s)}{w(s)}\int_0^\pi \psi_2(t)\,w(t)f(t)\,dt + \ldots + \frac{\psi_n(s)}{w(s)}\int_0^\pi \psi_n(t)\,w(t)f(t)\,dt + \ldots, \quad (40)$$

where $s$ is the point of $(0, \pi)$ corresponding to the point $x$ of $(a, b)$.

In connection with the series just written, let us consider the series

$$\psi_1(s)\int_0^\pi \psi_1(t)f(t)\,dt + \psi_2(s)\int_0^\pi \psi_2(t)f(t)\,dt + \ldots + \psi_n(s)\int_0^\pi \psi_n(t)f(t)\,dt + \ldots . \quad (41)$$

This latter is a canonical Sturm-Liouville series corresponding to $f(s)$; and $f(s)$, it will be recalled, is $\mathrm{F}(x)$ expressed as a function of $s$. *We shall refer to* (41) *as the canonical Sturm-Liouville series related to the general Sturm-Liouville series* (39).

Let $\sigma_n(s)$ be the sum of the first $n$ terms of the series (41); and let $\overline{\sigma}_n(s)$ be the sum of the first $n$ terms of the series (40). We proceed to show that, as $n$ increases indefinitely,

$$\sigma_n(s) - \overline{\sigma}_n(s)$$

converges to zero uniformly in the whole of $(0, \pi)$.

In the first place, let us suppose that the pair of boundary conditions satisfied by the functions $\Psi_n(x)$ is B; the normal functions $\psi_n(s)$ will therefore satisfy B′. By the result obtained in § 10, it is known that, as $n$ increases indefinitely,

$$\sigma_n(s) - \mathfrak{s}_n(s)$$

converges uniformly to zero in $(0, \pi)$; and by the results obtained in §§ 13, 14 (*cf.* § 21)

$$\mathfrak{s}_n(s) = \frac{1}{2\pi}\int_0^\pi [f_1(s-t) + f_1(s+t) + f_1(-s-t) + f_1(-s+t)]\,\frac{\sin\frac{1}{2}(2n-1)\,t}{\sin\frac{1}{2}t}\,dt.$$

Thus we see that

$$\sigma_n(s) - \frac{1}{2\pi}\int_0^{\pi}[f_1(s-t)+f_1(s+t)+f_1(-s-t)+f_1(-s+t)]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt$$

converges uniformly to zero in $(0, \pi)$.

It may be shown in a similar way that

$$\bar{\sigma}_n(s) - \frac{1}{2\pi w(s)}\int_0^{\pi}[h_1(s-t)+h_1(s+t)+h_1(-s-t)+h_1(-s+t)]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt$$

has the same property, the function $h_1(s)$ being defined for all values of $s$ by the rules

$$\begin{aligned} h_1(s) &= w(s)f(s) \quad (0 \leq s \leq \pi), \\ &= 0 \quad (-\pi < s < 0), \\ h_1(s+2\pi) &= h_1(s). \end{aligned}$$

It follows at once that, in order to establish the result stated, it will be sufficient to show that, as $n$ increases indefinitely, each of the four integrals

$$\int_0^{\pi}\left[\frac{h_1(\pm s \pm t)}{w(s)} - f_1(\pm s \pm t)\right]\frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t}\,dt \quad . \quad . \quad . \quad . \quad . \quad (42)$$

converges uniformly to zero in $(0, \pi)$.

§ 23. The function $w(s)$ is defined in the interval $(0, \pi)$ only, and, by the hypothesis of III., § 1, has a continuous differential coefficient in this interval. Let us define it for values of $s$ in $(-\pi, 0)$ in any manner consistent with the conditions (1) that in $(-\pi, \pi)$ it shall be always positive, (2) that it shall have a continuous differential coefficient in $(-\pi, \pi)$, and (3) that $w(-\pi) = w(\pi)$, $w'(-\pi) = w'(\pi)$;* then let us define it for values of $s$ outside $(-\pi, \pi)$ by the rule

$$w(s+2\pi) = w(s)$$

The function $w(s)$ defined in this way assumes only positive values, and has a differential coefficient which is everywhere continuous. Also, on referring to the definitions of the functions $f_1(s)$, $h_1(s)$, it will be seen that

$$h_1(s) = w(s)f_1(s),$$

for all values of $s$.

* One method of doing this is as follows: Let $w_1(s)$ be the (possibly, a) rational integral cubic function of $s$ whose coefficients are such that $w_1(0) = w(0)$, $w_1'(0) = w'(0)$, $w_1(-\pi) = w(\pi)$, $w_1'(-\pi) = w'(\pi)$. Since $w_1(-\pi)$ and $w_1(0)$ are both positive, we can clearly choose a number C great enough to ensure that

$$w_2(s) = w_1(s) + Cs^2(\pi+s)^2$$

is positive in the whole of $(-\pi, 0)$. The conditions (1), (2) and (3) will all be satisfied if we define $w(0)$ to be equal to $w_2(s)$ in $(-\pi, 0)$.

Consider now the integral

$$\int_0^\pi \left[\frac{h_1(s+t)}{w(s)} - f_1(s+t)\right] \frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t} dt.$$

It is equal to

$$\int_0^\pi f_1(s+t) \left[\frac{w(s+t)}{w(s)} - 1\right] \frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t} dt;$$

and this in turn is equal to

$$\int_0^\pi f_1(s+t)\,\chi(s,t)\sin\tfrac{1}{2}(2n-1)t\,dt, \quad \ldots \ldots \ldots \quad (43)$$

where

$$\chi(s,t) = \frac{w(s+t)-w(s)}{w(s)\sin\frac{1}{2}t} \quad (0 < t \leq \pi), \qquad = \frac{2w'(s)}{w(s)} \quad (t = 0).$$

Recalling that $w(s)$ is never zero, it is evident that $\chi(s,t)$ will be a continuous function of $s$ and $t$ in the square $0 \leq s \leq \pi$, $0 \leq t \leq \pi$, if the function

$$\chi_1(s,t) = \frac{w(s+t)-w(s)}{t} \quad (t \neq 0), \qquad = w'(s) \quad (t = 0),$$

is everywhere continuous. The only points at which there can be any doubt as to the continuity of $\chi_1(s,t)$ are those which lie on $t = 0$; and it is not very difficult to see that the function is continuous at these points also. For, when $t \neq 0$, we have

$$\chi_1(s+\eta,t) - \chi_1(s,0) = \frac{w(s+\eta+t)-w(s+\eta)}{t} - w'(s) \qquad = w'(s+\eta+\theta t) - w'(s),$$

where, by the mean value theorem, $0 < \theta < 1$. Since $w'(s)$ is continuous we see at once that

$$\lim_{\eta \to 0\, t \to 0} \chi_1(s+\eta,t) = \chi_1(s,0),$$

which proves that $\chi_1(s,t)$ is a continuous function of $s$ and $t$ at points on the line $t = 0$. It is therefore clear that $\chi(s,t)$ is continuous in the square $0 \leq s \leq \pi$, $0 \leq t \leq \pi$; and hence, in virtue of the second corollary of II., § 9, that (43) converges uniformly to zero in $(0, \pi)$, as $n$ increases indefinitely. It follows that

$$\int_0^\pi \left[\frac{h_1(s+t)}{w(s)} - f_1(s+t)\right] \frac{\sin\frac{1}{2}(2n-1)t}{\sin\frac{1}{2}t} dt$$

converges uniformly to zero in $(0, \pi)$.

It may be shown in the same way that each of the other integrals (42) has this property. As we have already seen, this is sufficient to establish that

$$\sigma_n(s) - \bar{\sigma}_n(s)$$

converges uniformly to zero in $(0, \pi)$.

Hitherto it has been assumed that the functions $\Psi_n(x)$ satisfy the pair of boundary conditions B. By using the results of §§ 11, 12, and following the line of proof indicated above, it will be found that the final result is unaffected when the pair of boundary conditions happens to be either ${}_a\text{B}$, ${}^b\text{B}$, or ${}^b_a\text{B}$. Hence, as $\overline{\sigma}_n(s)$ is the sum of the first $n$ terms of the series (39), we have the theorem :—

*The limits of indeterminacy of a Sturm-Liouville series at any point are the same as the limits of indeterminacy of the canonical Sturm-Liouville series related to it at the corresponding point of* $(0, \pi)$. *Further, if the former series converges uniformly in any set of points, the canonical series related to it converges uniformly in the corresponding set of points of* $(0, \pi)$.

§ 24. The theorem of the preceding paragraph enables us to translate the theorems of §§ 17–21 into theorems on the convergence of the Sturm-Liouville series

$$\Psi_1(x)\int_a^b g(y)\,\Psi_1(y)\,\text{F}(y)\,dy+\Psi_2(x)\int_a^b g(y)\,\Psi_2(y)\,\text{F}(y)\,dy$$

$$+\ldots+\Psi_n(x)\int_a^b g(y)\,\Psi_n(y)\,\text{F}(y)\,dy+\ldots \quad . \quad . \quad (39)$$

As a preliminary we recall that, if $s$ is the point of $(0, \pi)$ which corresponds to $x$ of $(a, b)$,

$$s = \xi\int_a^x\left(\frac{g}{k}\right)^{1/2}dx.$$

It follows that the point $s-t$ of $(0, \pi)$ corresponds to $x-y$, where, replacing $t$ by $y$ for convenience of notation, $y_1$ is the function of $x$ and $y$ defined by

$$y = \xi\int_{x-y_1}^x\left(\frac{g}{k}\right)^{1/2}dx\,;$$

further, the point $s+t$ of $(0, \pi)$ corresponds to the point $x+y_2$, where

$$y = \xi\int_x^{x+y_2}\left(\frac{g}{k}\right)^{1/2}dx.$$

The functions $y_1$, $y_2$ evidently have positive values for positive values of $y$, and tend to zero with $y$.

Referring now to the results of § 17, it will be evident that the limits of indeterminacy of the series (41), at a point of the open interval $(0, \pi)$, are

$$\overline{\lim_{n\to\infty}}\,\frac{1}{2\pi}\int_0^\alpha[f(s-t)+f(s+t)]\,\frac{\sin\frac{1}{2}(2n-1)\,t}{\sin\frac{1}{2}t}\,dt,$$

for values of $\alpha$ which are sufficiently small. Since $f(s)$ is $\text{F}(x)$ expressed as a function of $s$, it follows that the limits of indeterminacy of (39) at the corresponding point of $(a, b)$ are

$$\overline{\lim_{n\to\infty}} \frac{1}{2\pi}\int_0^\alpha [\mathrm{F}(x-y_1)+\mathrm{F}(x+y_2)] \frac{\sin\frac{1}{2}(2n-1)y}{\sin\frac{1}{2}y}\,dy.$$

Hence the theorem :—

*At any particular point of the open interval* $(a, b)$ *the limits of indeterminacy of the Sturm-Liouville series* (39) *depend only upon the values assumed by* $\mathrm{F}(x)$ *and* $\frac{g}{k}$ *in an arbitrarily small neighbourhood of the point.*

In a similar way it may be shown that :—

*At an end-point of* $(0, \pi)$, *where the functions* $\Psi_n(x)$ *do not satisfy the boundary condition* $v = 0$, *the limits of indeterminacy of the Sturm-Liouville series* (39) *depend only upon the values assumed by* $\mathrm{F}(x)$ *and* $\frac{g}{k}$ *in an arbitrarily small neighbourhood to the right, or left, of the point, as the case may be.*

The statement of the theorem which corresponds to the third of § 17 is left to the reader.

§ 25. When $\mathrm{F}(x)$ is of limited total fluctuation in an interval of $(a, b)$, the function $f(s)$ is of limited total fluctuation in the corresponding interval of $(0, \pi)$; also, if $x$ is any point of the former interval, and $s$ is the corresponding point of the latter, we have

$$\mathrm{F}(x-0) = f(s-0), \quad \mathrm{F}(x+0) = f(s+0).$$

It follows at once, from the second theorem of § 18, that :—

*If* $\mathrm{F}(x)$ *has limited total fluctuation in an arbitrarily small neighbourhood of a point* $x$ *belonging to the open interval* $(a, b)$, *the Sturm-Liouville series* (39) *converges, and has the sum* $\frac{1}{2}[\mathrm{F}(x-0)+\mathrm{F}(x+0)]$ *at this point.*

Again, we see from III., § 24, that when $\Omega(x)$ exists, $w(s)$ exists and is equal to it. Hence, from the third theorem of § 18, we see that :—

*At any point* $x$ *of the open interval* $(a, b)$, *where* $\Omega(x)$, *the bilateral limit of* $\mathrm{F}(x)$, *exists as a finite number, a sufficient condition that the Sturm-Liouville series* (39) *may converge, and therefore have* $\Omega(x)$ *for its sum, is that*

$$\left|\frac{\mathrm{F}(x-y_1)+\mathrm{F}(x+y_2)-2\Omega(x)}{y}\right|$$

*should have a Lebesgue integral with respect to* $y$ *in an interval* $(0, \alpha)$.

The corollary to this theorem which corresponds to that of § 18 is of interest, since it may be stated without the intervention of the functions $y_1$ and $y_2$. It will be found to read as follows :—

*At any point* $x$ *of the open interval* $(a, b)$, *where* $\mathrm{F}(x-0)$ *and* $\mathrm{F}(x+0)$ *exist as* *finite numbers, a sufficient condition that the Sturm-Liouville series* (39) *may converge, and therefore have* $\frac{1}{2}[\mathrm{F}(x-0)+\mathrm{F}(x+0)]$ *for its sum, is that*

$$\left|\frac{\mathrm{F}(x-y)-\mathrm{F}(x-0)}{y}\right|, \qquad \left|\frac{\mathrm{F}(x+y)-\mathrm{F}(x+0)}{y}\right|$$

*should both have Lebesgue integrals with respect to* $y$ *in an interval* $(0, \alpha)$.

There is no necessity to give the analogues of the theorems of §§ 19, 20, for their statement can present no difficulty. The reader should observe, however, that, corresponding to the particular case mentioned in the enunciation of the last theorem of § 20, we have a criterion of uniform convergence which does not involve the functions $y_1$ and $y_2$. It is, as follows:—

*A sufficient condition that the Sturm-Liouville series* (39) *may converge uniformly in an interval* $(a_1, b_1)$ *lying within* $(a, b)$ *is that*

$$\int_0^\beta\left|\frac{\mathrm{F}(x-y)-\mathrm{F}(x)}{y}\right|dy, \qquad \int_0^\beta\left|\frac{\mathrm{F}(x+y)-\mathrm{F}(x)}{y}\right|dy$$

*should exist as Lebesgue integrals, for each value of* $x$ *in* $(a_1, b_1)$, *and that, as* $\beta$ *diminishes indefinitely, both should converge uniformly to zero in* $(a_1, b_1)$.

§ 26. We saw in § 23 that the difference

$$\sigma_n(s)-\bar{\sigma}_n(s)$$

converges uniformly to zero in the whole of $(0, \pi)$. From the lemma of II., § 10, it follows that the difference between the arithmetic mean of the first $n$ partial sums of the series

$$\frac{\psi_1(s)}{w(s)}\int_0^\pi \psi_1(t)\,w(t)f(t)\,dt+\frac{\psi_2(s)}{w(s)}\int_0^\pi \psi_2(t)\,w(t)f(t)\,dt+\ldots+\frac{\psi_n(s)}{w(s)}\int_0^\pi \psi_n(t)\,w(t)f(t)\,dt+\ldots \quad (40)$$

and the arithmetic mean of the first $n$ partial sums of

$$\psi_1(s)\int_0^\pi \psi_1(t)f(t)\,dt+\psi_2(s)\int_0^\pi \psi_2(t)f(t)\,dt+\ldots+\psi_n(s)\int_0^\pi \psi_n(t)f(t)\,dt+\ldots \quad . \quad (41)$$

converges uniformly to zero, as $n$ increases indefinitely.

Recalling that the terms of (40) are equal to the corresponding terms of

$$\Psi_1(x)\int_a^b g(y)\,\Psi_1(y)\,\mathrm{F}(y)\,dy+\Psi_2(x)\int_a^b g(y)\,\Psi_2(y)\,\mathrm{F}(y)\,dy$$
$$+\ldots+\Psi_n(x)\int_a^b g(y)\,\Psi_n(y)\,\mathrm{F}(y)\,dy+\ldots, \quad . \quad (39)$$

and applying results proved above (§ 21) in regard to the convergence of the arithmetic mean of the first $n$ partial sums of (41), we obtain the theorems:—

*The arithmetic mean of the first* $n$ *partial sums of the Sturm-Liouville series* (39) *converges to* $\lim_{y \to 0} \frac{1}{2}[F(x-y_1)+F(x+y_2)]$, *as n increases indefinitely, at each point of the open interval* $(a, b)$ *where this limit exists as a finite number; moreover, at a point where this limit has one of the improper values* $\pm \infty$, *it diverges to this value and is non-oscillatory. If the set of points at which* $F(x)$ *is continuous includes a closed interval lying within* $(a, b)$, *then the arithmetic mean converges to* $F(x)$ *uniformly in this interval.*

*If the functions* $\Psi_n(x)$ *do not satisfy the boundary condition* $v = 0$ *at* $\begin{matrix} x = a \\ x = b \end{matrix}$, *then, as n increases indefinitely, the arithmetic mean of the first n partial sums of the Sturm-Liouville series* (39) *converges at* $\begin{matrix} x = a \\ x = b \end{matrix}$ *to* $\begin{matrix} F(a+0) \\ F(b-0) \end{matrix}$, *whenever this limit exists as a finite number; moreover, when this limit has one of the improper values* $\pm \infty$, *the arithmetic mean diverges to this value and is non-oscillatory.*



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